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***Symmetric polynomials***

Seminar in mathematics Entitled:

***Introduction :***

All us dealt with relationships between variables likeand

Which stays the same if we made a kind of permutation on them without even thinking of their name.

That what we call symmetric polynomials …

But have we ever asked ourselves:

***What is special about symmetric polynomials ???***

***Are they necessarily homogenous ???***

***Are there any theorems and properties about them that may help us to solve problems ???***

***Is there any relationship between symmetric polynomials and triangle geometry ???***

***And finally , how can we use symmetric to solve problems and prove theorems and relationships ???***

That what we will study in our research …...

***1.What are symmetric polynomials ??***

***1.1What is definition of a symmetric polynomial ??[[1]](#footnote-2)***

Before defining a symmetric polynomial we should know before what is a polynomial.

A polynomial is an expression consisting a sum of a finite number of terms each term is the product of a constant coefficient and one or more variables are raised to non-negative integer power . Like the polynomial with two variables and is a sum of terms of the form ,where andare non-negative integers and is a constant ,and the degree of the term is the sum of ,and the degree of the polynomial is equal to the term with the highest degree ,and we denote it by

For Example

A polynomial with three variables is a sum of terms of the form , whereare non-negative integers and is a constant ,and the degree of each term is the sum of ,and the degree of the polynomial is equal to the degree of the term with the largest value of ,and we denote the polynomial as

***Example :***

And in general , a polynomial with n variables is the sum of terms with the form C where is a non-negative integer and the degree of each term is given by and the degree of the polynomial is equal to the degree of the term with the highest degree .

Then we can define a symmetric polynomial as a polynomial which remains unchanged if its variables are permuted .

A symmetric polynomial with two variables andsatisfies :

And a symmetric polynomial with three variables satisfies :

And here are some examples for symmetric polynomials :

**(1)**is a symmetric polynomial of the degree 3.

**(2)**is a symmetric polynomial of the degree 4.

(**3) The sum of**  is a symmetric polynomial of the degree .

**(4)** is a symmetric polynomial of the degree ,where n is a non-negative integer .

**(5),**is a polynomial of the degree 9 .

***1.2 Are symmetric polynomials necessarily homogeneous?[[2]](#footnote-3)***

As we said before ,a symmetric polynomial doesn't change while its variables are being permuted, whereas a homogeneous polynomial is a polynomial such that all the terms have the same degree.

So ,we find that a symmetric polynomial isn't necessarily homogeneous ,and a homogeneous polynomial isn't necessarily symmetric .

In examples (1) and (5) in the previous part, the polynomials were symmetric but not homogeneous .

While the polynomials in examples (2) ,(3) and (4) was symmetric and homogeneous at the same time .

The polynomial : is homogeneous but it is not symmetric.

***1.3 Vieta's family and symmetric polynomials :[[3]](#footnote-4)***

If we have the following polynomial :

And are the roots of this polynomial .

Then :

Each one of is a member of viete's family .

And in general , the polynomial is called the symmetric polynomial because it has the following property :

For all permutations of [[4]](#footnote-5)

***2. What are the basic results and theorems in symmetric polynomials that may help us to solve problems ??***

All results ,theorems in this chapter holds for symmetric polynomials with n variables ,but maybe in some of them we will restrict ourselves to polynomials with three variables then we may generate them .

***2.1 The Fundamental theorem :***[[5]](#footnote-6)

***The Fundamental theorem***: For any symmetric polynomial F() ,there exists a polynomial F() which is not necessarily symmetric such that :

***Proof :***

Before proving this theorem ,there is something we have to prove then we become able to prove this theorem .

In general ,if we have two monomials with the variables of the same degree,then we write :

Where are non-negative integers

Which means that is earlier than , if or if and

So , if we have a symmetric polynomial whose monomials are all of the same degree , then there is a monomial among them which is earlier than the others , we shall call it the leading monomial of

***For example:***

The leading monomial for this polynomial is .

***Lemma 1 :*** A monomial is a leading monomial of a symmetric polynomial if and only if :

. And if then we can say that the monomial

is the leading monomial of .

Let us consider that then so the polynomial has a monomial

 which is earlier than .So cannot be the leading monomial , and if ,then there is a monomial which is earlier than ,so , also cannot be the leading monomial of the polynomial

.

And for the second part of the lemma : The leading part of is equal to the product of the leading monomials of , that is the product of

Now let us prove the main theorem which is The Fundamental theorem:

Let be the leading monomial of then and is also the leading monomial of the polynomial

So ,by doing the process:

we will get a symmetric polynomial whose leading monomial is later than

We can continue this process (of reducing of the leading monomials) , and since we have only a finite number of a fixed degree ,finally we shall obtain the required presentation of

***In general ,***[[6]](#footnote-7)any symmetric polynomial in variables with coefficients in can be written as a ( not necessarily a symmetric ) polynomial of the

That is :

***Examples :***Write the following symmetric polynomials as polynomials of

and from the previous example we found that :

***2.2 Muirhead's theorem :[[7]](#footnote-8)***

Let us consider the set of sequences which has the form , where that satisfies :

*1)*

*2)*

Let and be two -tuples from such that

then we say that majoriesif :

***Example :* (**

We write :

Which means that is equal to the sum of the terms by permutation of the

***For example :***

If then :

For

**(4**

Then we can write Muirhead's theorem which states that :

***Muirhead's theorem :***For all non-negative , then the following inequality holds :

And the equality holds if ,or if all the are equal .

***Proof :***

We will proof this theorem for then the general case can be proved the same way .

Suppose that and not all the are equal .

We will consider the following three partial cases then the general case will follow:

 Let and be a positive real number such that and then :

And we know that :

Observe that the two differences and are positive together or negative together and is also positive ,so there product is positive .So, the right hand side is positive and since not all the are equal ( , the inequality holds .

and then :

And we know :

The two differences and are both positive or both negative ,so the right hand side is positive ,and not all the are equal ,then the inequality

holds .

And the same way we can prove that if ,then for any and

Assume that .Then , and Then by ,

, where =As

 , hence by (b). Thus as required . The case

is dealt with similarly using and . If , the statement follows straight from .

***Example :***For any non-negative numbers the following inequality holds :

***2.3 Newton's formula for power sums :****[[8]](#footnote-9)*

Let be a polynomial which is the sum of the powers ( (which is a polynomial of :symmetric

Where are the roots of :

Then :

***3.Trianglegeometry and symmetric polynomials ….***

***-Is there any relationship between triangle geometry and symmetric equations ??***

The answer is yes ,there is many relationships between triangle geometry and symmetric functions ,and they are very useful for solving problems and proving theorems ,and the next theorems show us some of the relationships between triangles and symmetric equations .

 ***3.1 Cubic equations and triangle geometry :***

***Theorem(1)[[9]](#footnote-10) :*** If and are the roots of a cubic equation :

Then it's obvious from Vieta's theorem and the fundamental theorem that any symmetric function of the roots and is a polynomial function of the coefficients

The value := is known as the discriminant of the equation above .

Let us write it in terms of :

=

***Theorem(2)[[10]](#footnote-11):*** 1) Let be real then has exactly three real roots of and only if the discriminant of the equation is non-negative ,that means :

2) The roots of the equation are the lengths of a triangle if and only if :

 , which means that all these roots are positive .

***Proof :***To prove the first part of the theorem ,we should prove that the first side leads to the second one and the second side leads to the first one .

It’s obvious that if the roots of the equation are real hen :

=

Now , if , we know that either all the roots and are real or one of them is real and the other two roots are complex numbers , we will show that the second case is not possible ,suppose that is real , ,then :

Where , but we know that , so the second case can't occur .

For the second part of the theorem :

If

And

And

Then it's clear that

Either all the factors are positive and then we can construct a triangle or two factors are negative ,let us consider that and , by adding these two inequalities we get and it's impossible .

***3.2 Relationships between lengths of sides of a triangle and its semperimeter ,inradius and circumradius .[[11]](#footnote-12)***

If we have a triangle and and are the lengths of its sides respectivily,

then there are relationships that connect and with and ,where is semperimeter, is the inradius and is the circumradius .

And the most common relationships are :

The first one is the definition of ,and the third one follows from the area of a triangle

So

Where means the area of the triangle

Now the second relationship :

We know that :

Now we can substitute and in

***Result :***Any symmetric polynomial of and can be written as polynomial in terms of ,so it can be also expressed as a polynomial in and .

***Examples :***If are the lengths of sides of a triangle and and are the, semperimeter the inradius and the circumradius of the trianglerespetivily .

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 Also , if we have a cubic equation , and the lengths of the sides of a triangle are the roots of this equation then we can write it down :

Where  is semperimeter, is the inradius and  is the circumradius of the triangle  .

***Theorem(3)[[12]](#footnote-13):***if we have three positive numbers then we say that they are the circumradius ,the inradius and the semiperimeter respectively of a triangle if and only if :

***Proof :*** To prove that the circumradius ,inradius and semiperimeter of a triangle, we have to show that the roots of the cubic equation ( and let them and  ) :

 Are the sides of a triangle.So , first we have to prove that the roots of this cubic are real and for that we will use the condition :

After some manipulating we will get the condition :

 .

Which is equal to the condition  .

  Then we have to show that the roots are positive which comes from the fact that   are positive numbers ,so :

So   and  are positive .

 And finally we can prove that the roots and  are the lengths of the sides of a triangle and we know that the positive roots  and  are the lengths of the sides of a triangle if and only if :

(

***4.How can we use symmetric polynomials to solve problems and prove relationships ??***

In this chapter ,we will discuss some examples which have symmetric polynomials or which can be solved using symmetric polynomials and its properties,and we will show you some techniques to solve them .

***Problem (1):***Solve the equation

***Solution (1):***Suppose that  ,then the equation takes the form :

And we have :

By writing in terms of we get :

By substituting ,we get :

either :

or :

Then we can obtain the values of by finding the solutions of the systems :

***Problem (2) :*** What is the relation between  and  if :

***Solution (2) :***

***Problem (3) :***What conditions must satisfy and in order that are different positive real numbers and

***Solution (3) :***

Then

By substituting  ,we get :

Since all the numbers  and  should be positive then the sum of this numbers should be positive, too .That means

Now ,we get,and

Since we have :

Then  are the roots of the quadric equation :

But the two roots should be real and different :

So, the both differences between parentheses are positive together or negative together .

Since ,then

 So ,

***Problem (4) :*** Find integer solutions for the equation :

***Solution (4):***

Either :

Then , all the pairs of the form or ,where is an integer ,are solutions of this equation .

Or :

We can see that neither nor are possible for this equation .So, we can write :

Let us write:

Then is an integer ,if and only if divides .

So,

Then : .

***Problem (5):***Find all integers for which there are positive integers such that :

***Solution (5) :***Let us consider that is a fixed number and solutions of the equation are of the form where And let us choose from this set of solutions the one for which is the minimum , then either or .

If :

And because is a positive integer then :

either ,or

Now, if

Observe that a solution of the equation , which we can write it in another way :

And let be the other root ,then by Vieta's formulas :

Since ,

So , is a solution of the equation ,and since is the solution with for that is the minimum sum then

Which means that and that is impossible because is a positive integer .

So , and

***Problem (6) :***Solve the system :

, ,.[[13]](#footnote-14)

***Solution (6):***

Then are the solutions of the cubic equation :

[

Either

or

So finally we get .

***Problem (7) :*** Prove that in a triangle if and are the measurements of vertices of the triangle then :

***Solution (7) :*** We know that if we have a triangle and and are the lengths of the sides and respectively andis the measurement of the angle that faces then

=

=

***Problem (8) :*** Let a, b and c be the lengths of the sides of a triangle, prove that

***Solution (8) :***We know that ,where is semperimeter, is the inradius and is the circumradius of the triangle .

Then by substituting the value of and in the inequality, it takes the form :

Using the sine law which says that if are the lengths of the sides respectively in a triangle and are the measurements of the the angles that faces respectively then :

Then the inequality takes the form :

And we know that )

***Problem (9) :***Let a, b and c be the lengths of the sides of a triangle, prove that

***Solution (9) :***

Using the inequality between the harmonic men and the geometric mean ,we get :

We have :

***Problem (10) :***Let a, b and c be the lengths of the sides of a triangle. Using the tools we have, prove that :

***Solution (10) :*** From the previous problem ,we have :

Using inequality ,we get :

Using and we get :

***Problem (11):***Let and be the lengths ofthe sides of a triangle.If, prove

***Solution (11):***We have :

We know that ,where is semperimeter, is the inradius and is the circumradius of the triangle .

***Problem (12):***Let and be the lengths of the sides of a triangle and let be the circumradius, prove that :

 ***Solution (12) :***Let us consider that is the semperimeter and is the inradius of the triangle ,then

Using this fact we can write :

And we know that

There is a fact says that

***((Conclusion ))***

As we have seen , although symmetric polynomials subject a simple one ,they have a very big importance in mathematics either it was in solving problems or proving theorems and relationships in both Algebra and triangle geometry ,by making use of basic theorems and properties and knowing how to use them with the best techniques and ways ,depending on the kind of the problem and its information , so I hope that we give them more and more attention and make use of their properties and theorems because they help us to solve problems in a simple way .

And finally , I hope that this research admired you despite its simplicity , and I wish that I could spotlight this polynomials because of its important uses in the world of mathematics .

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