Gauss elimination

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10th grade

3rd section

Syrian Arab Republic

Ministry of Education

National Center for Distinguished

2015-2016

المحتويات والفهرس:

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# Introduction:

In our real life, we have a lot of problems to solve it. There are short ways and hard ways to solve it. So, in linear equations for example:

2x + y = 4 : x,y € N

y = x – 2

And the question is:

X = ? y = ?

Clearly, the answer is:

X = 2 → y = 0

But in other examples like:

X, Y, Z € N

4x + 2y + 6z =150

3x + 4z = 95

10z + 23y = 280

And the question is:

X = ? y = ? z = ?

It’s hard to know or it takes a lot of time, so, is there any way to solve it in a short period and easier way?

And if I want to program it, didn’t it cross the timeline? In other word, is it easy to program it?

# Who’s Gauss?

Johann Carl Friedrich Gauss, He was born in 30 April 1777 and died in 23 February 1855. He is sometimes referred to as the "Prince of Mathematicians" and the "greatest mathematician since antiquity". He has had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians. He named the mathematics as the queen of the sciences or the mother of the science.

He was born in Bronshfige in Germany. His parents was poor. His mother was ignorance and she don't know his birthday, but she remembered that he was born in Wednesday. But she knew that he was born before the festival of the place of ascent by eight days, later, he solved this problem and specify his birthday. He was very intelligent and he did a lot of discoveries in mathematic (when he was a child).

## His discoveries and intelligent!

At just three years old, he corrected an error in his father payroll calculations, and he was looking after his father’s accounts on a regular basis by the age of 5. At the age of 7.

A story about him

In mid ages and in a remote village, there was a child named as Gauss, he was very intelligent, clever and distinguished, so every time his teacher asked him a question, he was the first one to answer and his friends haven't any chance to answer because he was thinking too fast. Once upon a time, the teacher asked gauss a hard question to summon the numbers from 1 to 100, quickly he answered 5050.

The way he thought

He said: I find that 1 + 99 = 100 and 2 + 98 = 100 and 3 + 97 = 100 …. 49 + 51 = 100 and 100

So I have a 50 couple of numbers that equal 100 (with number 100) and 50

So the total is 50 \* 100 + 50 = 5050.

The teacher was amazed by his genius.

Although his family was poor and working class, Gauss' intellectual abilities attracted the attention of the Duke of Brunswick, who sent him to the Collegium Carolinum at 15, and then to the prestigious University of Göttingen (which he attended from 1795 to 1798). It was as a teenager attending university that Gauss discovered (or independently rediscovered) several important theorems.

At 15, Gauss was the first to find any kind of a pattern in the occurrence of prime numbers, a problem which had exercised the minds of the best mathematicians since ancient times. Although the occurrence of prime numbers appeared to be almost competely random, Gauss approached the problem from a different angle by graphing the incidence of primes as the numbers increased. He noticed a rough pattern or trend: as the numbers increased by 10, the probability of prime numbers occurring reduced by a factor of about 2 (e.g. there is a 1 in 4 chance of getting a prime in the number from 1 to 100, a 1 in 6 chance of a prime in the numbers from 1 to 1,000, a 1 in 8 chance from 1 to 10,000, 1 in 10 from 1 to 100,000, etc). However, he was quite aware that his method merely yielded an approximation and, as he could not definitively prove his findings, and kept them secret until much later in life.

In Gauss’s annus mirabilis of 1796, at just 19 years of age, he constructed a hitherto unknown regular seventeen-sided figure using only a ruler and compass, a major advance in this field since the time of [Greek](http://www.storyofmathematics.com/greek.html) mathematics, formulated his prime number theorem on the distribution of prime numbers among the integers, and proved that every positive integer is representable as a sum of at most three triangular numbers.

Although he made contributions in almost all fields of mathematics, number theory was always Gauss’ favourite area, and he asserted that “mathematics is the queen of the sciences, and the theory of numbers is the queen of mathematics”. An example of how Gauss revolutionized number theory can be seen in his work with complex numbers (combinations of real and imaginary numbers).

And he a lot of incredible things for the society and mathematics.

# Gauss theories

## Schoolbook Elimination:

Some time was needed to develop the concept of equations [**Heeffer**, 2011], and even then, of 107 algebras printed between 1550 and 1660 in the late Renaissance, only four books had *simultaneous* linear equations [**Kloyda**, 1938].  
The earliest example found by Kloyda was from Jacques Peletier du Mans [1554]. He solved a problem of Girolamo Cardano to find the money held  
by three men when each man’s amount plus a fraction of the others’ is given. **Peletier** first took the approach of **Cardano**. This solution was by  
verbal reasoning in which the symbolic algebra is a convenient shorthand. The discourse has variables for just two amounts, and it represents the third by the given formula of the other two. **Peletier** [p. 111] then re-solved the problem almost as we do, starting from three variables and equations  
and using just symbolic manipulation (restated here with modern symbols):  
***2R + A + B = 64  
R + 3A + B = 84  
R + A + 4B = 124  
2R + 4A + 5B = 208  
3A + 4B = 144  
3R + 4A + 2B = 148  
3R + 2A + 5B = 188  
6R + 6A + 7B = 336  
6R + 6A + 24B = 744  
17B = 408***  
Peletier’s overlong calculation suggests that removing unknowns systematically was a further advance. That step was soon made by Jean Borrel, who wrote in Latin as Johannes **Buteo** [1560, p. 190]. Borrel and the *Nine Chapters* both used the same double-multiply elimination process   
(restated with modern symbols):  
***3A + B + C = 42***  
***A + 4B + C = 32  
A + B + 5C = 40  
11B + 2C = 54  
2B + 14C = 78  
150C = 750***Lecturing on the algebra in Renaissance texts became the job of Isaac Newton upon his promotion to the Lucasian professorship. In  
1669–1670 Newton wrote a note saying that he intended to close a gap in the algebra textbooks:  
“This bee omitted by all that have writ introductions to this Art, yet I judge it very proper & necessary to make an introduction complete” [Whiteside, 1968–1982, v. II, p. 400, n. 62]. Newton’s contribution lay unnoticed for many years until his notes were published in Latin in 1707 and then in English in 1720. Newton stated the recursive strategy for solving simultaneous equations whereby one equation is used to remove a variable  
from the others.  
And you are to know, that by each Equation one unknown Quantity may be taken away, and consequently, when there are as many  
Equations and unknown Quantities, all at length may be reduced into one, in which there shall be only one Quantity unknown.  
— Newton [1720, pp. 60–61] Newton meant to solve any simultaneous algebraic equations. He included rules to remove one variable from two equations which need not be linear: substitution (solve an equation for a variable and place the formula in the other) and equality-of values (solve in both and set the formulas equal).

While Newton’s notes awaited publication, Michel Rolle [1690, pp. 42ff.] also explained how to solve simultaneous, specifically linear, equations. He arranged the work in two columns with strict patterns of substitutions. We may speculate that Rolle’s emphasis survived in the “method of  
substitution” and that his “column du retour” is remembered as “backward” substitution. Nevertheless, Newton influenced later authors more  
strongly than Rolle.  
In the eighteenth century many textbooks appeared “all more closely resembling the algebra of Newton than those of earlier writers” [**Macomber,**  
1923, p. 132]. Newton’s direct influence is marked by his choice of words. He wrote “exterminate” in his Latin notes [**Whiteside**, 1968–1982, v. II, p. 401,  
no. 63] that became “exterminate” in the English edition and the derivative texts. A prominent example is the algebra of Thomas Simpson [1755, pp.  
63ff.]. He augmented Newton’s lessons for “the Extermination of unknown quantities” with the rule of addition and/or subtraction (linear combination  
of equations).  
Among many similar algebras, Sylvestre Lacroix made an important contribution to the nomenclature. His polished textbooks presented the best material in a consistent style [**Domingues**, 2008], which included a piquant name for each concept. Accordingly, **Lacroix** [1804, p. 114] wrote, “This operation, by which one of the unknowns is removed, is called *elimination*” (Cette opération, par laquelle on chasse une des inconnues, senomme *élimination*). The first algebra printed in the United States was a translation by John Farrar of Harvard College [Lacroix, 1818]. As derivative texts were written, “this is called elimination” became a fixture of American algebras.  
Gaussian elimination for the purpose of school books was thus complete by the turn of the nine teenth century. It was truly *schoolbook* elimination, because it had been developed to provide readily undertaken exercises in symbolic algebra.

## Matrix Interpretation:

The milieu of using symbolic algebra to modify Gaussian elimination ended with the adoption of matrix algebra. Several authors had developed matrices in the second half of the nineteenth century [Hawkins, 1975, 1977a, b, 2008]. Although matrices were not needed to compute by hand, the new representational technology showed that all the proliferating elimination algorithms were trivially related through matrix decompositions. Eventually matrices would help organize calculations for the purpose of programming electronic computers.  
This development leads from the astronomical observatory of the Jagiellonian University to the numerical analysis textbooks that are presently in your campus bookstore. Manual computing motivated astronomer Tadeusz **Banachiewicz** [1938a,b] to independently invent matrices in the form called **Cracovians**. They have a column-bycolumn product, which is the natural way to calculate with columns of figures by hand.  
It must, however, be conceded that in practice it is easier to multiply column by column than to multiply row by column …. It may, in  
fact, be said that the computations are made by **Cracovians** and the theory by matrices.

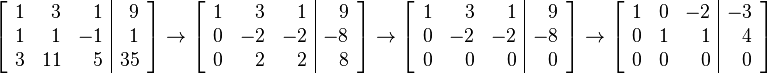
Banachiewicz advocated using Cracovians to represent calculations as early as 1933. The idea realized in Arthur Cayley’s matrix algebra by  
two people. Henry Jensen [1944] of the Danish Geodætisk Institut used pictograms, = , to emphasize that three algorithms for solving normal equations amounted to expressing a square matrix as a triangular product: the “Gaussian algorithm” (the calculation with Gauss’s  
brackets), the Cracovian method, and Cholesky’s method. A noteworthy aspect of Jensen’s presentation was suggested by Frazer et al. [1938]: to  
represent arithmetic operations through multiplication by “elementary matrices”. In the same year Paul Dwyer [1944] of the University  
of Michigan showed that Doolittle’s method was an “efficient way of building up” some “so-called triangular” matrices. He found no similar interpretation except in the work of Banachiewicz. Thecoincident papers of Jensen and Dwyer are the earliest to depict Gaussian elimination in roughly  
the modern form, that is, in terms of Cayleyan matrices.  
A deep use for the matrix interpretation came from John von Neumann and his collaborator Herman Goldstine. They and others were in the process of building the first programmable, electronic computers. Concerns over the efficacy of the machines motivated von Neumann and Golds tine to study Gaussian elimination. The initial part of their analysis introduced the matrix decomposition.  
We may therefore interpret the elimination method as … the combination of two tricks:  
First, it decomposes *A* into a product of two [triangular] matrices … [and second] it forms their inverses by a simple, explicit, inductive process.  
— von Neumann and Goldstine [1947] Von Neumann and Goldstine used matrix algebra to establish bounds on the rounding errors of what they anticipated would be the mechanized algorithm once computers became available. When the matrix is symmetric and positive definite, their  
bound remains the best that has been achieved. Although Gaussian elimination is observed to be accurate, a comparable error bound has yet to be established in the general case.6  
The next step to the campus bookstore was aided by *Mathematical Reviews*. John Todd found Jensen’s paper through *MR* and lectured on it at King’s College London [Taussky and Todd, 2006]. An auditor communicated the matrix formalism to staff at the National Phys  
ical Laboratory. Among them was Alan Turing, who evidently learned of the matrix interpretation both from Jensen through Todd  
and also during a visit to von Neumann and Gold stine, whom Turing [1948] cited. He described Gaussian elimination in the manner of von Neumann and Goldstine by treating the general case of schoolbook elimination and in the manner of Jensen with elementary matrices. Turing wrote  
with a brevity of expression that made ideas clear without overworking them.  
The invention of electronic computers created a discipline that was at first populated by those who made scientific calculations [Traub, 1972;  
Wilkinson, 1970]. Among them, George Forsythe was a visionary mathematician who is reputed to have named “computer science”  
[Knuth, 1972]. Gauss’s involvement lent credence to the subject matter of the new discipline. The terminology that geodesists had used to describe  
the calculations of Gauss suggested an origin for what then was named simply “elimination”. In an address to the American Mathematical Society,  
Forsythe [1953] misattributed “high school” elimination to Gauss and appears to have been the first to call it “Gaussian elimination” [Grcar, 2011a, tab.1]. The name was widely used within a decade.  
The university mathematics curriculum adopted matrix descriptions more slowly. Linear algebra itself was not commonly taught until the 1960s.  
When Fox [1964] and Forsythe and Moler [1967] wrote influential numerical analysis textbooks that featured the matrix interpretation, then they  
reprised Turing’s presentation.

Coda**:**   
An *algorithm* is a series of steps for solving a mathematical problem. The matrix interpretation of Gaussian elimination seldom becomes an algorithm in a straightforward way, because the speed of computing depends on whether the calculation is well adapted to the problem and the  
computer. Just as Gauss developed the first professional method for least-squares calculations and then Doolittle developed a method for use  
with multiplication tables, other methods were developed more recently to solve the equations of finite-element analysis [Irons, 1970] with parallel  
computers [Duff and Reid, 1983]. While Cholesky and Crout emphasized sums of products for calculating machines, the arithmetic steps can be reordered automatically to suit different computer architectures [Whaley and Dongarra, 1998].  
More radical transformations are possible that reduce the work to solve *n* equations below O*(n*3*)* arithmetic operations [Strassen, 1969; Cohn and  
Umans, 2003; Demmel et al., 2007]. Perhaps the only certainty about future algorithms is their name. Rather than being a Platonic archetype,  
Gaussian elimination is an evolving technique.

# Gaussian elimination

In [linear algebra](https://en.wikipedia.org/wiki/Linear_algebra), Gaussian elimination (also known as row reduction) is an [algorithm](https://en.wikipedia.org/wiki/Algorithm) for solving [systems of linear equations](https://en.wikipedia.org/wiki/System_of_linear_equations). It is usually understood as a sequence of operations performed on the associated [matrix](https://en.wikipedia.org/wiki/Matrix_%28mathematics%29) of coefficients. This method can also be used to find the [rank](https://en.wikipedia.org/wiki/Rank_%28linear_algebra%29) of a matrix, to calculate the [determinant](https://en.wikipedia.org/wiki/Determinant) of a matrix, and to calculate the inverse of an [invertible square matrix](https://en.wikipedia.org/wiki/Invertible_matrix). The method is named after [Carl Friedrich Gauss](https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss) (1777–1855), although it was known to Chinese mathematicians as early as 179 CE (see [History section](https://en.wikipedia.org/wiki/Gaussian_elimination#History)).

To perform row reduction on a matrix, one uses a sequence of [elementary row operations](https://en.wikipedia.org/wiki/Elementary_row_operations) to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible. There are three types of elementary row operations: 1) Swapping two rows, 2) Multiplying a row by a non-zero number, 3) Adding a multiple of one row to another row. Using these operations, a matrix can always be transformed into an [upper triangular matrix](https://en.wikipedia.org/wiki/Triangular_matrix), and in fact one that is in [row echelon form](https://en.wikipedia.org/wiki/Row_echelon_form). Once all of the leading coefficients (the left-most non-zero entry in each row) are 1, and in every column containing a leading coefficient has zeros elsewhere, the matrix is said to be in [reduced row echelon form](https://en.wikipedia.org/wiki/Reduced_row_echelon_form). This final form is unique; in other words, it is independent of the sequence of row operations used. For example, in the following sequence of row operations (where multiple elementary operations might be done at each step), the third and fourth matrices are the ones in row echelon form, and the final matrix is the unique reduced row echelon form.



Using row operations to convert a matrix into reduced row echelon form is sometimes called Gauss–Jordan elimination. Some authors use the term Gaussian elimination to refer to the process until it has reached its upper triangular, or (non-reduced) row echelon form. For computational reasons, when solving systems of linear equations, it is sometimes preferable to stop row operations before the matrix is completely reduced.

## Definitions and example of algorithm

The process of row reduction makes use of [elementary row operations](https://en.wikipedia.org/wiki/Elementary_row_operations), and can be divided into two parts. The first part (sometimes called Forward Elimination) reduces a given system to *row echelon form*, from which one can tell whether there are no solutions, a unique solution, or infinitely many solutions. The second part (sometimes called [back substitution](https://en.wikipedia.org/wiki/Triangular_matrix#Forward_and_back_substitution)) continues to use row operations until the solution is found; in other words, it puts the matrix into *reduced* row echelon form.

Another point of view, which turns out to be very useful to analyze the algorithm, is that row reduction produces a [matrix decomposition](https://en.wikipedia.org/wiki/Matrix_decomposition) of the original matrix. The elementary row operations may be viewed as the multiplication on the left of the original matrix by [elementary matrices](https://en.wikipedia.org/wiki/Elementary_matrix). Alternatively, a sequence of elementary operations that reduces a single row may be viewed as multiplication by a [Frobenius matrix](https://en.wikipedia.org/wiki/Frobenius_matrix). Then the first part of the algorithm computes an [LU decomposition](https://en.wikipedia.org/wiki/LU_decomposition), while the second part writes the original matrix as the product of a uniquely determined invertible matrix and a uniquely determined reduced row echelon matrix.

### ***Row operations***

There are three types of elementary row operations which may be performed on the rows of a matrix:

***Type 1***: Swap the positions of two rows.

***Type 2***: Multiply a row by a nonzero [scalar](https://en.wikipedia.org/wiki/Scalar_%28mathematics%29).

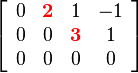
***Type 3***: Add to one row a scalar multiple of another.

If the matrix is associated to a system of linear equations, then these operations do not change the solution set. Therefore, if one's goal is to solve a system of linear equations, then using these row operations could make the problem easier.

### ***Echelon form***

For each row in a matrix, if the row does not consist of only zeros, then the left-most non-zero entry is called the [*leading coefficient*](https://en.wikipedia.org/wiki/Leading_coefficient) (or *pivot*) of that row. So if two leading coefficients are in the same column, then a row operation of type 3 (see [above](https://en.wikipedia.org/wiki/Gaussian_elimination#Row_operations)) could be used to make one of those coefficients zero. Then by using the row swapping operation, one can always order the rows so that for every non-zero row, the leading coefficient is to the right of the leading coefficient of the row above. If this is the case, then matrix is said to be in row echelon form. So the lower left part of the matrix contains only zeros, and all of the zero rows are below the non-zero rows. The word "echelon" is used here because one can roughly think of the rows being ranked by their size, with the largest being at the top and the smallest being at the bottom.

For example, the following matrix is in row echelon form, and its leading coefficients are shown in red.



It is in echelon form because the zero row is at the bottom, and the leading coefficient of the second row (in the third column), is to the right of the leading coefficient of the first row (in the second column).

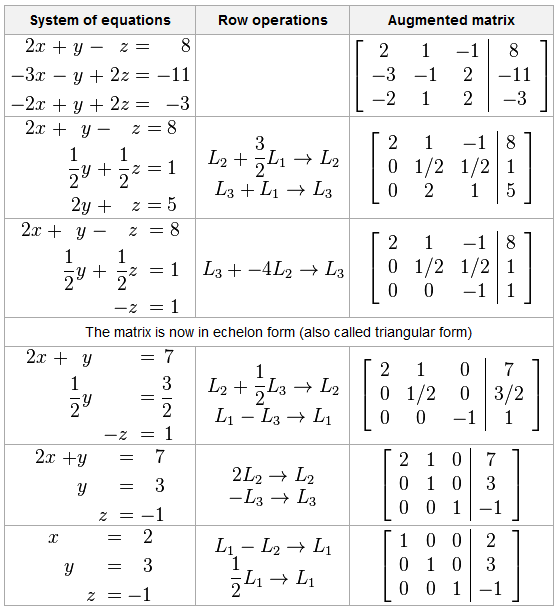
A matrix is said to be in reduced row echelon form if furthermore all of the leading coefficients are equal to 1 (which can be achieved by using the elementary row operation of type 2), and in every column containing a leading coefficient, all of the other entries in that column are zero (which can be achieved by using elementary row operations of type 3).

### ***Example of the algorithm***

Suppose the goal is to find and describe the set of solutions to the following [system of linear equations](https://en.wikipedia.org/wiki/System_of_linear_equations):

\begin{alignat}{7}
2x &&\; + \;&& y             &&\; - \;&& z  &&\; = \;&& 8 & \qquad (L_1) \\
-3x &&\; - \;&& y             &&\; + \;&& 2z &&\; = \;&& -11 & \qquad (L_2) \\
-2x &&\; + \;&& y &&\; +\;&& 2z  &&\; = \;&& -3 &  \qquad (L_3)
\end{alignat}

The table below is the row reduction process applied simultaneously to the system of equations, and its associated [augmented matrix](https://en.wikipedia.org/wiki/Augmented_matrix). In practice, one does not usually deal with the systems in terms of equations but instead makes use of the augmented matrix, which is more suitable for computer manipulations. The row reduction procedure may be summarized as follows: eliminate *x* from all equations below L_1, and then eliminate *y* from all equations below L_2. This will put the system into [triangular form](https://en.wikipedia.org/wiki/Triangular_form). Then, using back-substitution, each unknown can be solved for.



The second column describes which row operations have just been performed. So for the first step, the *x* is eliminated from (L2) by adding (3/2L1) to (L2). Next *x* is eliminated from (L3) by adding (L1) to (L3). These row operations are labelled in the table as

* + C:\Users\HP\Desktop\New folder (4)\Gaussian elimination - Wikipedia, the free encyclopedia_files\0caf3e4fe4ed7835271b7b12f1eb2264.png → C:\Users\HP\Desktop\New folder (4)\Gaussian elimination - Wikipedia, the free encyclopedia_files\d843d71869425cb9eefb67d52e20c972.png

C:\Users\HP\Desktop\New folder (4)\Gaussian elimination - Wikipedia, the free encyclopedia_files\7fe29c3b56f8256d6155df6d58909af3.png

Once *y* is also eliminated from the third row, the result is a system of linear equations in triangular form, and so the first part of the algorithm is complete. From a computational point of view, it is faster to solve the variables in reverse order, a process known as back-substitution. One sees the solution is z = -1, y = 3, and x = 2. So there is a unique solution to the original system of equations.

Instead of stopping once the matrix is in echelon form, one could continue until the matrix is in *reduced* row echelon form, as it is done in the table. The process of row reducing until the matrix is reduced is sometimes referred to as Gauss-Jordan elimination, to distinguish it from stopping after reaching echelon form.

## History

The method of Gaussian elimination appears in the Chinese mathematical text [Chapter Eight *Rectangular Arrays*](https://en.wikipedia.org/wiki/Rod_calculus#System_of_linear_equations) of [*The Nine Chapters on the Mathematical Art*](https://en.wikipedia.org/wiki/The_Nine_Chapters_on_the_Mathematical_Art). Its use is illustrated in eighteen problems, with two to five equations. The first reference to the book by this title is dated to 179 CE, but parts of it were written as early as approximately 150 BCE. [[1]](#footnote-1)[[2]](#footnote-2)It was commented on by [Liu Hui](https://en.wikipedia.org/wiki/Liu_Hui) in the 3rd century.

The method in Europe stems from the notes of [Isaac Newton](https://en.wikipedia.org/wiki/Isaac_Newton). In 1670, he wrote that all the algebra books known to him lacked a lesson for solving simultaneous equations, which Newton then supplied. Cambridge University eventually published the notes as *Arithmetica Universalis* in 1707 long after Newton left academic life. The notes were widely imitated, which made (what is now called) Gaussian elimination a standard lesson in algebra textbooks by the end of the 18th century. [Carl Friedrich Gauss](https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss) in 1810 devised a notationfor symmetric elimination that was adopted in the 19th century by professional [hand computers](https://en.wikipedia.org/wiki/Human_computer) to solve the normal equations of least-squares problems. The algorithm that is taught in high school was named for Gauss only in the 1950s as a result of confusion over the history of the subject.[[5]](https://en.wikipedia.org/wiki/Gaussian_elimination#cite_note-5)[[3]](#footnote-3)

Some authors use the term *Gaussian elimination* to refer only to the procedure until the matrix is in echelon form, and use the term Gauss-Jordan elimination to refer to the procedure which ends in reduced echelon form. The name is used because it is a variation of Gaussian elimination as described by [Wilhelm Jordan](https://en.wikipedia.org/wiki/Wilhelm_Jordan_%28geodesist%29) in 1887. However, the method also appears in an article by Clasen published in the same year. Jordan and Clasen probably discovered Gauss–Jordan elimination independently.[[6]](https://en.wikipedia.org/wiki/Gaussian_elimination#cite_note-6)

## Applications

The historically first application of the row reduction method is for solving [systems of linear equations](https://en.wikipedia.org/wiki/Systems_of_linear_equations). Here are some other important applications of the algorithm.

### ***Computing determinants***

To explain how Gaussian elimination allows the computation of the determinant of a square matrix, we have to recall how the elementary row operations change the determinant:

* Swapping two rows multiplies the determinant by -1
* Multiplying a row by a nonzero scalar multiplies the determinant by the same scalar
* Adding to one row a scalar multiple of another does not change the determinant.

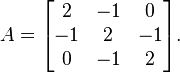
If the Gaussian elimination applied to a square matrix *A* produces a row echelon matrix *B*, let *d* be the product of the scalars by which the determinant has been multiplied, using above rules. Then the determinant of *A* is the quotient by *d* of the product of the elements of the diagonal of *B*: det(*A*) = ∏diag(*B*) / *d*.

Computationally, for a *n*×*n* matrix, this method needs only [*O*(*n*3)](https://en.wikipedia.org/wiki/O_notation) arithmetic operations, while solving by elementary methods requires *O*(2*n*) or *O*(*n*!) operations. Even on the fastest computers, the elementary methods are impractical for *n* above 20.

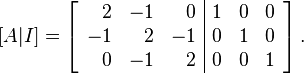
### ***Finding the inverse of a matrix***

A variant of Gaussian elimination called Gauss–Jordan elimination can be used for finding the inverse of a matrix, if it exists. If *A* is a *n* by *n* square matrix, then one can use row reduction to compute its [inverse matrix](https://en.wikipedia.org/wiki/Invertible_matrix), if it exists. First, the *n* by *n* [identity matrix](https://en.wikipedia.org/wiki/Identity_matrix) is augmented to the right of *A*, forming a *n* by 2*n* [block matrix](https://en.wikipedia.org/wiki/Block_matrix) [*A* | *I*]. Now through application of elementary row operations, find the reduced echelon form of this *n* by *2n* matrix. The matrix *A* is invertible if and only if the left block can be reduced to the identity matrix *I*; in this case the right block of the final matrix is *A*−1. If the algorithm is unable to reduce the left block to *I*, then *A* is not invertible.

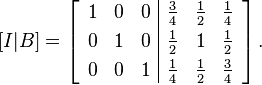
For example, consider the following matrix



To find the inverse of this matrix, one takes the following matrix augmented by the identity, and row reduces it as a 3 by 6 matrix:



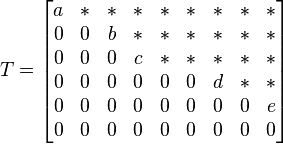
By performing row operations, one can check that the reduced row echelon form of this augmented matrix is:



One can think of each row operation as the left product by an [elementary matrix](https://en.wikipedia.org/wiki/Elementary_matrix). Denoting by *B* the product of these elementary matrices, we showed, on the left, that *BA* = *I*, and therefore, *B* = *A*−1. On the right, we kept a record of *BI* = *B*, which we know is the inverse desired. This procedure for finding the inverse works for square matrices of any size.

### ***Computing ranks and bases***

The Gaussian elimination algorithm can be applied to any m \times nmatrix A. In this way, for example, some 6 \times 9matrices can be transformed to a matrix that has a row echelon form like



where the \*s are arbitrary entries and *a, b, c, d, e* are nonzero entries. This echelon matrix Tcontains a wealth of information about A: the [rank](https://en.wikipedia.org/wiki/Rank_of_a_matrix) of Ais 5 since there are 5 non-zero rows in T; the [vector space](https://en.wikipedia.org/wiki/Vector_space) spanned by the columns of Ahas a basis consisting of the first, third, fourth, seventh and ninth column of A(the columns of *a, b, c, d, e* in T), and the \*s tell you how the other columns of Acan be written as linear combinations of the basis columns. This is a consequence of the distributivity of the [dot product](https://en.wikipedia.org/wiki/Dot_product) in the expression of a linear map [as a matrix](https://en.wikipedia.org/wiki/Linear_map#Matrices).

All of this applies also to the reduced row echelon form, which is a particular row echelon form.

## Computational efficiency

The number of arithmetic operations required to perform row reduction is one way of measuring the algorithm's computational efficiency. For example, to solve a system of *n* equations for *n* unknowns by performing row operations on the matrix until it is in echelon form, and then solving for each unknown in reverse order, requires *n*(*n*+1) / 2 divisions, (2*n*3 + 3*n*2 − 5*n*)/6 multiplications, and (2*n*3 + 3*n*2 − 5*n*)/6 subtractions,[[7]](https://en.wikipedia.org/wiki/Gaussian_elimination#cite_note-7) for a total of approximately 2*n*3 / 3 operations. Thus it has [arithmetic](https://en.wikipedia.org/wiki/Arithmetic#Arithmetic_operations) complexity of O(*n*3); see [Big O notation](https://en.wikipedia.org/wiki/Big_O_notation). This arithmetic complexity is a good measure of the time needed for the whole computation when the time for each arithmetic operation is approximately constant. This is the case when the coefficients are represented by [floating point numbers](https://en.wikipedia.org/wiki/Floating_point_number) or when they belong to a [finite field](https://en.wikipedia.org/wiki/Finite_field). If the coefficients are [integers](https://en.wikipedia.org/wiki/Integer) or [rational numbers](https://en.wikipedia.org/wiki/Rational_number) exactly represented, the intermediate entries can grow exponentially large, so the [bit complexity](https://en.wikipedia.org/wiki/Bit_complexity) is exponential. However, there is a variant of Gaussian elimination, called [Bareiss algorithm](https://en.wikipedia.org/wiki/Bareiss_algorithm) that avoids this exponential growth of the intermediate entries, and, with the same arithmetic complexity of O(*n*3), has a bit complexity of O(*n*5).

This algorithm can be used on a computer for systems with thousands of equations and unknowns. However, the cost becomes prohibitive for systems with millions of equations. These large systems are generally solved using [iterative methods](https://en.wikipedia.org/wiki/Iterative_method). Specific methods exist for systems whose coefficients follow a regular pattern (see [system of linear equations](https://en.wikipedia.org/wiki/System_of_linear_equations)).

To put an *n* by *n* matrix into reduced echelon form by row operations, one needs n^3arithmetic operations; which is approximately 50% more computation steps.[[4]](#footnote-4)

One possible problem is [numerical instability](https://en.wikipedia.org/wiki/Numerical_stability), caused by the possibility of dividing by very small numbers. If, for example, the leading coefficient of one of the rows is very close to zero, then to row reduce the matrix one would need to divide by that number so the leading coefficient is 1. This means any error that existed for the number which was close to zero would be amplified. Gaussian elimination is numerically stable for [diagonally dominant](https://en.wikipedia.org/wiki/Diagonally_dominant) or [positive-definite](https://en.wikipedia.org/wiki/Positive-definite_matrix) matrices. For general matrices, Gaussian elimination is usually considered to be stable, when using [partial pivoting](https://en.wikipedia.org/wiki/Pivot_element#Partial_and_complete_pivoting), even though there are examples of stable matrices for which it is unstable. [[5]](#footnote-5)

### ***Generalizations***

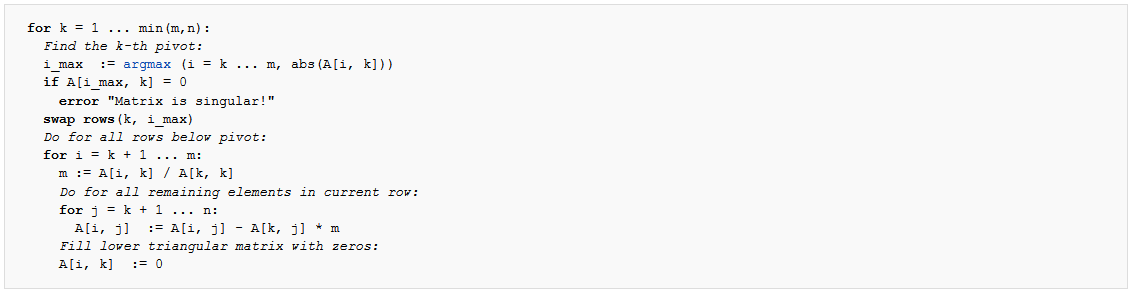
The Gaussian elimination can be performed over any [field](https://en.wikipedia.org/wiki/Field_%28mathematics%29), not just the real numbers.

Gaussian elimination does not generalize in any simple way to higher order [tensors](https://en.wikipedia.org/wiki/Tensors) (matrices are [array](https://en.wikipedia.org/wiki/Array_data_structure) representations of order 2 tensors); even computing the rank of a tensor of order greater than 2 is a difficult problem.

## Pseudocode

As explained above, Gaussian elimination writes a given m× nmatrix Auniquely as a product of an invertible m× mmatrix Sand a row-echelon matrix T. Here, Sis the product of the matrices corresponding to the row operations performed.

The formal algorithm to compute Tfrom Afollows. We write A[i,j]for the entry in row i, column jin matrix Awith 1 being the first index. The transformation is performed *in place*, meaning that the original matrix Ais lost and successively replaced by T.



This algorithm differs slightly from the one discussed earlier, because before eliminating a variable, it first exchanges rows to move the entry with the largest [absolute value](https://en.wikipedia.org/wiki/Absolute_value) to the [pivot](https://en.wikipedia.org/wiki/Pivot_element) position. Such *partial pivoting* improves the [numerical stability](https://en.wikipedia.org/wiki/Numerical_stability) of the algorithm; some other variants are used.

Upon completion of this procedure the augmented matrix will be in [row-echelon form](https://en.wikipedia.org/wiki/Row-echelon_form) and may be solved by back-substitution.

With modern computers, Gaussian elimination is not always the fastest algorithm to compute the row echelon form of matrix. There are [computer libraries](https://en.wikipedia.org/wiki/Library_%28computing%29), like [BLAS](https://en.wikipedia.org/wiki/BLAS), that exploit the specifics of the [computer hardware](https://en.wikipedia.org/wiki/Computer_hardware) and of the structure of the matrix to choose the best algorithm automatically.

As a conclusion, Gauss elimination is the best way to solve this linear equations and some of it may don’t be able to turn into program but to be more specified it’s harder to make it in time.

For the last question in the introduction:

X, Y, Z € N

4x + 2y + 6z = 150

3x + 4z = 95

10z + 3y = 80

And the question is:

X = ? y = ? z = ?

Answer:

**4x + 2y + 6z = 150**

**= 2x + y + 3z = 75 → y = 75 - 2x - 3z**

**So, 10z + 3 ( 75 – 2x – 3z ) = 80**

**10z + 225 – 6x – 9z = 80**

**Z – 6x = -145 (1)**

**3x + 4z = 95 (2)**

**3x + 4 (6x -145) = 95**

**3x + 24x – 580 = 95**

**27x = 675**

**X=25**

**Z=5**

**Y=10**

Answer with gauss:

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**2**[**Calinger** (1999)](https://en.wikipedia.org/wiki/Gaussian_elimination#CITEREFCalinger1999), pp. 234–236

**3** [**Grcar** (2011a)](https://en.wikipedia.org/wiki/Gaussian_elimination#CITEREFGrcar2011a), pp. 169-172, pp. 783-785, p. 789.

**4 J. B. Fraleigh** **and R. A. Beauregard**, Linear Algebra. Addison-Wesley Publishing Company, 1995, Chapter 10

**5** [**Golub & Van Loan** (1996)](https://en.wikipedia.org/wiki/Gaussian_elimination#CITEREFGolubVan_Loan1996), §3.4.6

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5. [Golub & Van Loan (1996)](https://en.wikipedia.org/wiki/Gaussian_elimination#CITEREFGolubVan_Loan1996), §3.4.6 [↑](#footnote-ref-5)