المركز الوطني للمتميزين
NATIONAL CENTER FOR THE DISTINGUISHED
/NCDSyria

A research entitled:

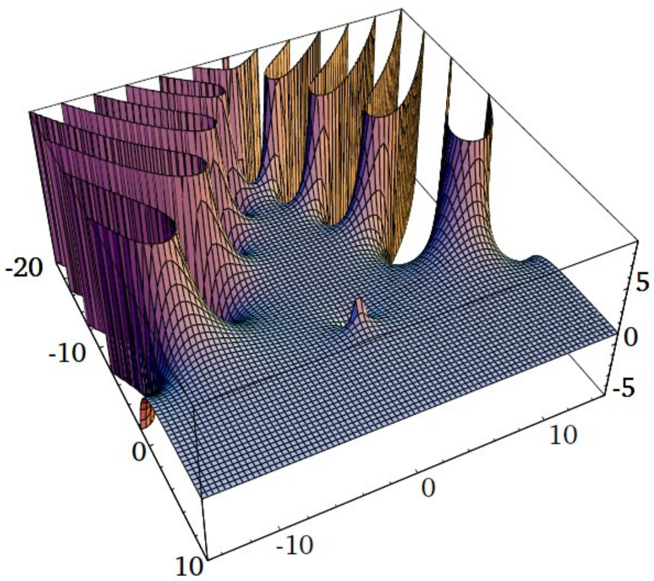
The Riemann hypothesis

Presented by: Ashraf Faraj

Grade: 12th

Year: 2016-2017

Supervised by: Habib Issa



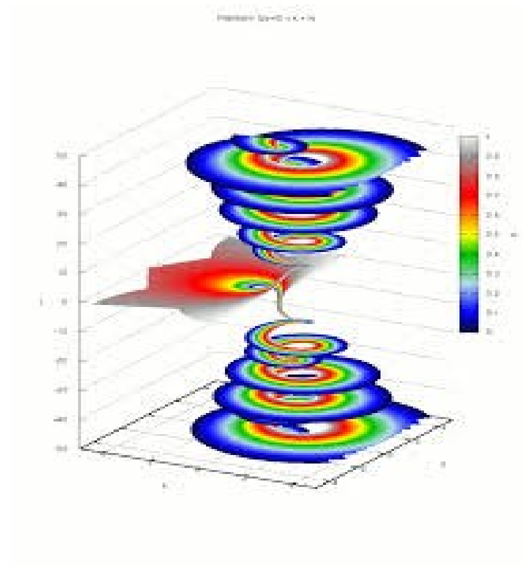


Table of contents

Introduction

The Riemann zeta function is a complex valued function defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for all s that satisfy $Re(s) > 1$ .This is obvious via the integral test, and because for $s = 1$, we find the well known harmonic series that is also divergent. Of course, Leonhard Euler was the first to study such function but only for real values and he was able to establish a tremendous result on finding the values of the function for all positive even numbers. He also found the following result, which will be of great use for us later on:

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^{-s}}\right)^{-1}$$

where the product is taken over all the primes. Euler's formula is thought to be as an analytic version of the fundamental theorem of arithmetic, which states that every number can be uniquely decomposed into a product of different prime powers. It also created a strong connection between the worlds of number theory and analysis paving the way in front of the appearance of analytic number theory. However, it was Bernhard Riemann who first studied the function as a complex valued function and then redefined it as a meromorphic function in the whole complex plane with the exception of a simple pole at $s = 1$ with residue 1 via analytic continuation, which we will show later on. After that, Riemann conjectured that the zeros of this function all lay on the line $Re(s) = 1/2$ .This conjecture is now known as the Riemann hypothesis, which will be the main topic of the following research. The Clay Mathematical institute placed the Riemann hypothesis as one of the seven problems of the millennium and proposed a one million dollar prize for the one who solves it. Therefore, I decided to write a research about the hypothesis for the following reasons:

1. Shedding light on the Riemann hypothesis being one of the most famous unsolved problems of the century.
2. Describing and choosing the best and most effective equivalences of the hypothesis.
3. Finding and deriving the computational strategies that are used to verify the hypothesis in certain intervals.
4. Finding some new equivalence based on Lagarias's equivalence.
5. Creating a powerful and consistent reference describing this hypothesis since I had difficulties in finding one.

Note: The following research assumes some knowledge with the basics of complex analysis and analytic number theory.

Chapter one: Generalities

1.1 Analytic continuation

The Riemann zeta function is defined by the following relation, which holds for all complex numbers, $s = \sigma + it$ such that $\sigma > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Actually, the former function can be extended to larger domains via a technique that we call analytic continuation, which is a basic technique in complex analysis that is used to extend the domains of analytic function form a certain domain to other larger ones.

First of all, we are going to establish an analytic continuation to the half plane $\sigma > 0$. Let's define the eta function $\eta(s)$, which is given by the relation:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}$$

We can easily check that the function is converging (i.e defined) for all $\sigma > 0$. We can easily observe that the following relations hold:

$$\begin{aligned}(1 - 2^{1-s})\zeta(s) &= \left(1 - 2.\frac{1}{2^s}\right)\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} \ldots\right) \\ &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} \ldots = \eta(s)\end{aligned}$$

Which means that:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s)$$

and that defines an analytic continuation to the semi plane $\sigma > 0$. [1]

Now we shall extend the zeta function to the whole complex plane with the exception of a zero at $s = 1$ with residue 1. First, let's recall the gamma function, which is given by:

$$\Gamma(s) = \int_0^{\infty} e^{-t}.t^{s-1}\,dt$$

which is valid for $Re(s) > 0$.

[1] Dwilewicz, R.J. and J. Minác, *Values of the Riemann zeta function at integers* , pages: 6.

By Weirestrass's factorization theorem, we find that:

$$\frac{1}{s.\Gamma(s)} = e^{\gamma s}.\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

which is an analytic continuation to the gamma function to the whole complex plane with the exception of simple poles at all negative integers. We have:

$$\Gamma\left(\frac{s}{2}\right) = \int_{0}^{\infty} e^{-t}.t^{\frac{s}{2}-1}\,dt$$

for $Re(s) > 0$. We put $t = n^2\pi x$ and we get:

$$\pi^{-\frac{s}{2}}.n^{-s}.\Gamma\left(\frac{s}{2}\right) = \int_{0}^{\infty} e^{-n^2\pi x}.x^{\frac{s}{2}-1}\,dx$$

by summing both sides according to n from 1 to infinity, and by changing the sum and the integral order (which is justified by the fact that the integral is convergent) we get:

$$\pi^{-\frac{s}{2}}.\zeta(s).\Gamma\left(\frac{s}{2}\right) = \int_{0}^{\infty}\left(\frac{\vartheta(x)-1}{2}\right).x^{\frac{s}{2}-1}\,dx$$

where:

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$$

this function is called the theta function and it satisfies a tremendous functional equation, which is valid for all $x > 0$: [2]

$$\sqrt{x}\vartheta(x) = \vartheta(1/x)$$

Using this functional equation, we arrive to the following shape:

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\left\{\frac{1}{s(s-1)} + \int_{1}^{\infty}\left(x^{\frac{s}{2}-1} + x^{\frac{s-2}{2}}\right).\left(\frac{\vartheta(x)-1}{2}\right)dx\right\}$$

Due to the exponential decay of the theta function, the improper integral on the right hand side is convergent for all s, but clearly, s can't be equal to one, which leads to theorem 1.

Theorem 1: The Riemann zeta function can be analytically continued to a meromorphic function with an exception of a simple pole at $s = 1$ with residue 1. [3]

[2] Segarra, E. *An Exploration of the Riemann Zeta Function and its Application to the Theory of Prime Number Distribution* , pages: 36.

1.2 The functional equation

To derive the functional equation of the zeta function, we notice that the expression between parentheses is unchanged under the substituting s by $1 - s$. which means that:

$$\zeta(s).\Gamma\left(\frac{s}{2}\right).\pi^{-s/2} = \zeta(1-s).\Gamma\left(\frac{1-s}{2}\right).\pi^{-(1-s)/2}$$

And right now, we define what some writers like to call "the complete zeta function":

$$\xi(s) = \zeta(s).\Gamma\left(\frac{s}{2}\right).\pi^{-s/2}$$

and with that definition, we see that the Riemann functional equation can be rewritten as:

$$\xi(s) = \xi(1-s)$$

and further, more we have that:

$$\overline{\xi(s)} = \xi(\bar{s})$$

which provides a further functional equation for the zeta function.

1.3 Values at integers

Values of the Riemann zeta function at integers are of great interest in many areas of science. But unfortunately, only values at even integers are known for this function. Some notes about the odd values will be discussed at the end of this section. Euler was the first to study values of the zeta function at integers and he obtained a relation giving the values of the zeta function at even integers in which we are going to derive right now using Lemma 1.

Theorem 2: The following relation holds for the even values of the Riemann zeta function:

$$\zeta(2k) = \frac{2^{2k-1}.\pi^{2k}.|B_{2k}|}{(2k)!}$$

where B_n is the n^{th} Bernoulli number.

Lemma 1: Let $f(z)$ be an analytic function in $\mathbb{C}$ with the exception of a finite number of points $z_1, z_2, z_3 \dots, z_k$ none of which is a real integer. Furthermore, let there exist a certain constant M such that $|z^2.f(z)| \leq M$ for all $|z| > p$ for some p. then the following holds:

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{j=1}^{k} R[\pi.\cot \pi z.f(z), z_j]$$

Now we are going to prove theorem 2.

---

[3] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, pages: 14.

Proof of theorem 2:

As we can see, the zeta function satisfies the conditions of the lemma except that it has a pole of order $2k$ at 0. This won't be a problem for us because now we put the zero in the list of the points at which the function is not analytic and we delete it from the summation interval. So this leaves us with the following relation:

$$\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1}{n^{2k}} = -R\left[\frac{\pi.\cot \pi z}{z^{2k}}, 0\right]$$

and because 0 is a pole of order $2k$ we calculate it by using the relation:

$$R\left[\frac{\pi.\cot \pi z}{z^{2k}}, 0\right] = \lim_{z\to 0}\frac{1}{(2k)!}\frac{d^{2k}}{dz^{2k}}\left(z^{2k+1}.\frac{\pi.\cot \pi z}{z^{2k}}\right)$$

To calculate the limit we shall present the following Taylor series expansion:

$$\pi z.\cot \pi z = B_0 + \sum_{n=1}^{\infty}(-1)^n.\frac{2^{2n}.B_{2n}.z^{2n}.\pi^{2n}}{(2n)!}$$

By derivation $2k$ times and then dividing by $2k$ factorial we get:

$$R\left[\frac{\pi.\cot \pi z}{z^{2k}}, 0\right] = \frac{2^{2k}.B_{2k}.\pi^{2K}}{(2k)!}$$

And because the sum on the left hand side of the relation is equal to twice the wanted sum we divide both sides on two to get the wanted relation (because the even Bernoulli numbers alter in signs the result is always a positive number). Concerning the odd negative integers we can find their value using the functional equation, which yields:

$$\zeta(1-2k) = 2.(2\pi)^{-2k}(-1)^k.(2k-1)!\,\zeta(2k)$$

There is no compact formula until now describing odd values of the Riemann zeta function. Furthermore, we don't know whether they are rational or not. Only the value at 3 was proven to be irrational by Apery and it was called Apery's constant. The following theorem (which is called Rivoal's theorem) is of great interest to give an insight about the nature of these constants.

Theorem 3: The sequence $\{\zeta(3), \zeta(5), \zeta(7), \zeta(9), \dots\}$ contains an infinite number of irrational numbers. More precisely the following estimate holds for the dimension $\delta(a)$ of the spaces generated over $\mathbb{Q}$ by $\{1, \zeta(3), \dots, \zeta(a-2), \zeta(a)\}$: [4]

$$\delta(a) \gg \frac{\ln a}{1+\ln 2}\big(1+o(1)\big)$$

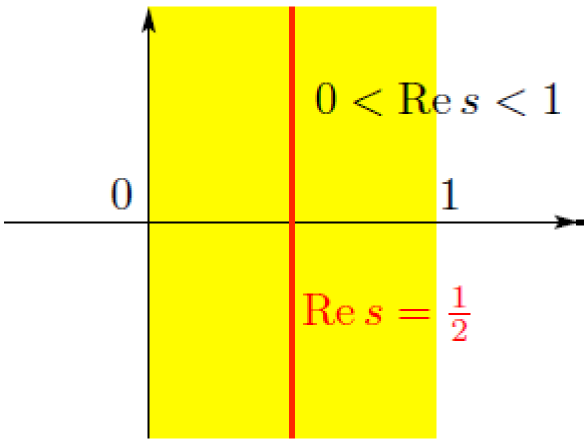
[4] Zudilin, W. *Irrationality of values of the Riemann zeta function*, page: 490.

1.4 The Riemann hypothesis

An important feature of every function is its zeros. So we tend to ask the following question: What are the zeros of the Riemann zeta function? and do they obtain any special properties. First of all from the Euler product formula that I showed in the introduction one can directly deduce that the zeta function has no zeros in the half plane $\sigma > 1$. Referring back to the analytic continuation of the zeta function to the complex plane, we find the gamma function lying in the denominator and because the gamma function has simple poles at negative integer values, we deduce that the zeta function has zeros at even negative integers. Those zeros are called the trivial zeros of the zeta function. But we are not concerned with those zeros. In his paper on number theory, Riemann conjectured that the zeta function has certain complex zeros of real part 1/2 which we know call non-trivial zeros and his conjecture is now known as the Riemann hypothesis which can be stated as follows:

The Riemann hypothesis: all the non-trivial zeros of the Riemann zeta function lie on the line $\sigma = 1/2$.[5]

And now we are going to state some terminology to use later on. The line that appears in the context of the hypothesis is called the critical line, which is shown in red below. And the region that lies between the lines $\sigma = 1$ and $\sigma = 0$ is called the critical strip and it's colored in yellow. And of course, in the next chapter an extensive study of the non-trivial zeros of the zeta function and an expansion of the zero-free region will be presented.



Picture 1: the critical line.

[5] Ghusayni, B. *Results connected to the Riemann hypothesis,* page: 1243,1244.

Chapter two: Properties of the Riemann zeta function

2.1 Zero-free region

A great result is that the Riemann zeta function has no zeros on the line $\sigma = 1$. Here I'm going to mention a proof of this result and talk a little bit about its consequences.

Theorem 1: $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$. [6]

Proof:

Recall that when $\sigma > 1$ the Riemann zeta function can be written as the following infinite product:

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Taking the natural logarithms of both sides of the earlier equation we have:

$$\ln \zeta(s) = -\sum_{\text{p prime}} \ln\left(1 - \frac{1}{\mathrm{p}^s}\right)$$

using the Taylor expansion of the function $\ln 1 - x$ around 0 we obtain:

$$\ln \zeta(s) = \sum_{p\ prime} \sum_{m=1}^{\infty} m^{-1}.p^{-sm}$$

which leads to:

$$\ln \zeta(s) = \sum_{p\ prime} \sum_{m=1}^{\infty} m^{-1}.p^{-\sigma m}.e^{-imt\ln p}$$

we can notice the following formula:

$$Re(\ln \zeta(s)) = \sum_{p\ prime} \sum_{m=1}^{\infty} m^{-1}.p^{-\sigma m}.\cos mt \ln p$$

assuming that $\tau_m = 3 + 4\cos mt \ln p + 3\cos 2mt \ln p$ and using the upper result we find:

$$3\mathrm{Re}(\ln \zeta(\sigma)) + 4\mathrm{Re}(\ln \zeta(\sigma + it)) + \mathrm{Re}(\ln \zeta(\sigma + 2it)) = \sum_{p\ prime} \sum_{m=1}^{\infty} \frac{1}{m.p^{\sigma m}}.\tau_m$$

then using $\ln z = \ln|z| + i.\arg(z)$ and the simple inequality $3 + 4\cos\theta + 3\cos 2\theta \geq 0$ we obtain:

---

[6] Gourdon, X. and P. Sebah, *Distribution of the zeros of the Riemann Zeta function*, page:2.

$$3|\ln\zeta(\sigma)|+4|\ln\zeta(\sigma+it)|+|\ln\zeta(\sigma+2it)|\geq 0$$

upon raising the exponential function to both the earlier sides the following inequality holds:

$$|\zeta(\sigma)|^3.|\zeta(\sigma+it)|^4.|\zeta(\sigma+2it)|\geq 1$$

since $\zeta(\sigma)$ has a simple pole at $\sigma=1$ with residue 1, its Laurent series at $\sigma=1$ has the form:

$$\zeta(\sigma)=\frac{1}{\sigma-1}+a_0+a_1(\sigma-1)+a_2(\sigma-1)^2\ldots.=\frac{1}{\sigma-1}+g(\sigma)$$

where the function g is analytic at $\sigma=1$. for $1<\sigma\leq 2$ we have that $g(\sigma)\leq A_0$ for some constant $A_0>0$, which gives us the following result:

$$|\zeta(\sigma)|\leq\frac{1}{\sigma-1}+A_0$$

Let's suppose that the function has a zero on the line 1. So $\zeta(1+it)=0$ for some $t\in\mathbb{R}^*$ and for some number $1<\sigma_0<\sigma$ by applying the mean value theorem we have:

$$|\zeta(\sigma+it)|=|\zeta(\sigma+it)-\zeta(1+it)|=(\sigma-1)\acute{\zeta}(\sigma_0+it)\leq(\sigma-1).A_1$$

where A_1 depends only on t. Also when $\sigma\to 1$ we have that $|\zeta(\sigma+it)|\leq A_2$ where this constant depends only on t. By using the earlier inequalities and taking the limit of as $\sigma\to 1^+$ we get the following result:

$$\lim_{\sigma\to 1^+}|\zeta(\sigma)|^3.|\zeta(\sigma+it)|^4.|\zeta(\sigma+2it)|$$

$$\leq\lim_{\sigma\to 1^+}\left(\frac{1}{\sigma-1}+A_0\right)^3.(\sigma-1)^4.{A_1}^4.A_2=0$$

which is an obvious contradiction leading to the result wanted. [7]

The earlier theorem expands the zero free region to be $0\leq\sigma<1$, and it's a base step in proving the PNT, which we are going to describe later on. We should also note that the best known extension of the zero free region is due to Korobov and Vinogradov. [8]

Theorem 2: For some constant $c>0$ there are no zeros of $\zeta(s)$ for $s=\sigma+it$ with large $|t|$ in the region:

$$1-\sigma\leq\frac{c}{(\ln|t|)^{3/2}.(\ln\ln|t|)^{1/3}}$$

[7] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, pages: 16,17 and 18.
[8] Christ, T. *Value-distribution of the Riemann zeta-function and related functions near the critical line*, page: 7.

2.2 Counting the number of zeros to a certain height

In his paper on the function, Riemann conjectured a certain formula to count the number of zeros to a certain height. The formula was then proved by Van Mangoldt and here I present a sketch of the proof. To see a full proof I refer the reader to[3].

Theorem 3: The number of zeros of the Riemann zeta function in the critical strip is given by the estimate:

$$N(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi} + O(\ln T)$$

Proof:

Let's define the following functions:

$$\theta(t) = \arg(\pi^{-\frac{it}{2}}.\Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)$$

$$S(t) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + it\right)$$

by applying the argument principle to the complete zeta function (since they both have the same zeros) we obtain the following result:

$$N(T) = 1 + \frac{\theta(T)}{\pi} + S(T)$$

then by using the asymptotic formula for the function $\theta(T)$ we have:

$$N(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

the function $S(T)$ satisfies the relation:

$$S(T) = O(\ln T)$$

as T tends to infinity. Which permits as to say that $N(T)$ behaves like $\frac{T}{2\pi}\ln\frac{T}{2\pi} + O(\ln T)$. So the zeros of the zeta function become denser and denser as one goes up in the critical strip. More precisely speaking, the ordinates of the n$^{\text{th}}$ zero of the zeta function behave like $2\pi n/\ln n$. The above results also allow us to state that the gaps between the zeros of the function are bounded. [9]

[9] Gourdon, X. and P. Sebah, *Distribution of the zeros of the Riemann Zeta function*, page: 3.

2.3 Hardy's theorem

Hardy's theorem was one of the strongest theorems that stand in the favor of the Riemann hypothesis. It states that there is infinitely many zeros of the zeta function on the critical line. As a quick note we see that the Riemann hypothesis does imply Hardy's theorem while the inverse is not true. For the proof, we need the following lemma, which is called Mellin's inversion formula.

Lemma 1: if:

$$F(s) = \int_{-\infty}^{+\infty} f(s).x^{-s}dx$$

then we have that:

$$f(s) = \int_{a-i\infty}^{a+i\infty} F(z).z^{s-1}dz$$

Theorem 4 (Hardy's theorem): The function $\zeta(s)$ has infinitely many zeros on the line 1/2.

Proof:

We are going to use the function $\xi(s)$. After some substitution and simplification we find:

$$\frac{2\xi(s)}{s(s-1)} = \int_{0}^{+\infty} u^{-s}\left(\vartheta(u^2) - 1 - \frac{1}{u}\right)du$$

applying the inversion formula for $0 < \sigma < 1$ we find that:

$$\vartheta(z^2) - 1 - \frac{1}{z} = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{2\xi(s)}{s(s-1)}ds$$

We should also note that the function $\vartheta(z^2)$ is defined (in addition for the real numbers) in the circular wedge W defined by $-\pi/4 < \arg(z) < \pi/4$. We claim that the earlier function and all its derivatives approach zero as z approaches $e^{i\pi/4}$ (e.g along the circle of unity). Using the functional equation we find (after a lot of simplification and manipulation):

$$\vartheta(z^2) = \frac{1}{(z^2-i)^{1/2}}\sum_{n\ odd} e^{\frac{-\pi n^2}{z^2-i}}$$

Since $u^k.e^{-1/u} \to 0$ as $u \to 0^+$ for any integer k, we prove our claim.

Considering the integral on the right hand-side of the equation, and using a previous relation we find:

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = \left|\frac{1/2+it}{1/2-it} - (1/2+it)\int_{1}^{\infty}\{x\}.x^{-3/2-it}dx\right| \ll |t|$$

as $t \to \pm\infty$. From Stirling's formula for the gamma function we find:

$$\ln\Gamma\left(\frac{1}{4}+\mathrm{i}\frac{\mathrm{t}}{2}\right) = \left(-\frac{1}{4}+\mathrm{i}\frac{\mathrm{t}}{2}\right)\ln\frac{1}{4}+\mathrm{i}\frac{\mathrm{t}}{2}-\frac{1}{2}-\mathrm{i}\frac{\mathrm{t}}{2}+\frac{1}{2}\ln 2\pi + O\left(\frac{1}{\mathrm{t}}\right)$$

it follows that:

$$\left|\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right| \ll e^{-(\pi/4)|t|}$$

as $t \to \pm\infty$. Thus we have:

$$\left|\xi\left(\frac{1}{2}+it\right)\right| \ll |t|^{3}e^{-(\pi/4)|t|}$$

which indicates that the earlier integral is convergent and hence analytic in the wedge W. By applying the operator $z.(d^2/dz^2).z$ then the formula takes a simpler form:

$$H(z) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} 2\xi(s).z^{s-1}ds$$

where $H(z)$ is $z.\frac{d^2\vartheta(z^2)}{dz^2}.z$. This shows that the function $H(z)$ and all its derivatives also approaches zero as $z \to e^{i\pi/4}$. Using the Taylor series of the exponential function we find:

$$\sqrt{z}.H(z) = \sum_{n=0}^{\infty} c_n(i\ln z)^n$$

where:

$$c_n = \frac{1}{\pi n!}\int_{-\infty}^{\infty}\xi(1/2+it).t^n$$

Let's assume that Hardy's theorem is wrong, then the function doesn't change sign except for a finite number of times which means that for large t $\xi(1/2+it)$ doesn’t change sign. If for example , $\xi(1/2+it)$ is positive for $t > T$ then we find:

$$c_{2n}.\pi.(2n)! \geq A_6(T+1)^{2n} - A_7T^{2n}$$

where A_6 and A_7 are positive constants. Thus c_{2n} is completely positive for sufficiently large n. The other way around is also true, thus if $\xi(1/2 + it)$ is negative for $t < T$, then c_{2n} is completely negative. So, if our function $\sqrt{z}.H(z)$ is differentiated sufficiently many times with respect to $i \ln z$ then we are going to have an even power series where all the coefficients have the same sign and it doesn't approach zero as $z \to e^{\pi i/4}$. But it must since $d/id \ln z = -iz\, d/dz$, and differentiating repeatedly creates a function that approaches zero as $z \to e^{\pi i/4}$ (as we proved earlier). This leads to a contradiction that proves Hardy's theorem. [10]

As we can see, Hardy's Theorem establishes a minimal and condition for the Riemann hypothesis. Selberg proved in 1950 that a positive portion of the zeros of the zeta function lies on the critical line. Levinson then proved that at least one third of the zeros lie on the line. The final and the best result known until now is that at least two fifths of the zeros lie on the critical line. The last result is due to Conrey. [11][4]

2.4: Vornin's universality theorem and its consequences

Through my study of the Riemann zeta function and its properties, I couldn't leave out the well known universality theorem. It states and with a great amount of intuition, that the Riemann zeta function is capable of approximating any analytic non-vanishing function in the strip $1/2 < \sigma < 1$ via vertical shifts in its argument. The theory was stated and proved by Vornin but here no proof will be mentioned. Consequences of the universality theory such as the functional independence of the zeta function shall be mentioned as well.

Theorem 5: Let $0 < r < 1/4$. Suppose that $f(s)$ is a continuous and non-vanishing on the disc $|s| \leq r$ and is analytic in the interior of the disc. Then for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ in which the following relation holds:

$$\max_{|s|\leq r} \left|\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon$$

Theorem 5 is considered to be an old version of the theory [12] .A modern version of the theorem has a bit more general form. Denote by $meas\{A\}$ the Lebseuge measure of the set A then the new version becomes.

Theorem 6: Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with a connected complement. Let $f(s)$ be a continuous, non-vanishing function on K, and analytic in the interior of K. Then, for every ε we have:

$$\lim_{T\to\infty} \inf \frac{1}{T} meas\{\tau \in [0, T] : \sup|\zeta(s + i\tau) - f(s)| < \varepsilon\} > 0$$

[10] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, pages: 25,26 and 27.
[11] Conrey, J.B. *More than two fifths of the zeros of the Riemann zeta function are on the critical line*, page: 1.
[12] Laurincikas, A. *Universality of the Riemann zeta function*, page: 2324.

The latter theorem states that the set of vertical shifts τ that satisfy the approximation property are fairly dense in the interval from 0 to infinity i.e it has a lower density that is bigger than zero. [13][5]

I shall also mention that the L- functions also satisfy the universality theorem. More on L functions will be discussed in chapter 4 when we talk about extensions of the Riemann hypothesis. One of the theoretical applications of the universality theorem is its application to the concept of functional independence. In 1887, Holder proved that the Euler gamma function doesn't satisfy any algebraic differential equation i.e there are no polynomials P that satisfy:

$$P\left(\Gamma(s), \acute{\Gamma}(s), \dots, \Gamma^{\mathrm{n}-1}(s)\right) = 0$$

In 1900,Hilbert observed that the Riemann zeta function also has the property of functional independence, which can be proved by the use of the result of Holder, and the functional equation, Vornin used his universality theorem to prove the independence of the Riemann zeta function. And the theorem can be stated as follows:

Theorem 7: Suppose that the functions $F_j : \mathbb{C}^n \to \mathbb{C}$ are continuous, $j = 0,1,2, \dots . r$ and:

$$\sum_{j=0}^{r} s^j . F_j(\zeta(s), \acute{\zeta}(s), \dots ., \zeta^{n-1}(s)) = 0$$

then $F_j = 0$ for all j between 0 and r.

One of the great application of the universality theorem is the ability to approximate analytic functions in a certain region. This can be done via the approximate functional equation, which can be used to approximate the Riemann zeta function. Here I state this approximate functional equation and I'm going to use it again in the next chapter.

Theorem 8: Suppose that $0 < \sigma < 1$ and suppose that $x, y, t \geq c > 0$ and $2\pi xy = t$. Then uniformly on σ we find:

$$\zeta(s) = \sum_{m \leq x} \frac{1}{m^s} + \chi(s) . \sum_{m \leq y} \frac{1}{m^{1-s}} + O(x^{-\sigma}) + O(t^{1/2-\sigma} . y^{\sigma-1})$$

where $\chi(s) = 2^s . \pi^{s-1} \Gamma(1-s) . sin \, \pi s/2$. This approximate functional equation with the help of the universality theorem, we can approximate any analytic non-vanishing function in any compact subset in the region $1/2 < \sigma < 1$. [14]

[13] Mauclaire, J.L. *Universality of the Riemann zeta function: two remarks,* pages: 313.
[14] Laurincikas, A. *The Riemann Zeta-function: Approximation of Analytic Functions*, pages: 111 and 112.

Chapter three: Computational strategies

3.1 The Euler Maclurin summation formula

Lemma 1: The Euler Maclurin summation formula in its most general version is given by the relation:

$$\sum_{n=N}^{M} f(n) = \int_{N}^{M} f(x)dx + \frac{1}{2}(f(N) + f(M)) + \sum_{n=1}^{v} \frac{B_{2n}}{(2n)!}.[f^{2n-1}(x)] + R_{2v}$$

where:

$$R_{2v} = \frac{-1}{(2v+1)!} \int_{N}^{M} B_{2v+1}(\{x\})\, f^{2v+1}(x)dx$$

$$[f(x)] = f(M) - f(N)$$

We will apply this result on the zeta function with some modification. If we take $N = 1$ and $M = \infty$ and $f(n) = 1/n^s$ and by knowing that the k[th] derivative of the function is given by:

$$f^k(s) = \frac{(-1)^k (s)(s+1)(s+2)\dots(s+k-1)}{n^{s+k}}$$

which means that:

$$[f^k(s)] = (-1)^k (s)(s+1)(s+2)\dots(s+k-1)$$

and then apply formula a great problem that will rise is that converge very slowly because of the choice of M and N. Instead, we might put $N = X$, $M = \infty$ and calculate the resulting series then we add the fixing term $\sum_{n=1}^{X} 1/n^s$ and by choosing the number X to be large enough we can easily compute the value of the zeta function at any argument i.e:

$$\zeta(s) = \sum\nolimits_{n=1}^{X} \frac{1}{n^s} + \frac{X^{1-s}}{1-s} + \frac{1}{2X^s} + \sum\nolimits_{n=1}^{v} \frac{B_{2n}}{(2n)!} \frac{\prod_{i=0}^{2(n-1)}(s+i)}{X^{s+2n-1}} + R_{2v}$$

where

$$R_{2v} = \frac{-s(s+1)(s+2)\dots(s+2v-1)}{(2v+1)!} . \int_{X}^{\infty} \frac{B_{2v+1}(\{x\})}{u^{s+2v}} du$$

The earlier summation is very useful in calculation but as we can see it might be very complicated on large scales, so better algorithms and methods shall be presented later on [15].

[15] Borwein, J.M., D.M. Bradley, and R.E. Crandall, *Computational strategies for the Riemann zeta function*, pages: 12 and 13.

3.2 Hardy's function

In order to present other computational strategies, a function of great interest called Hardy's function shall be presented. In order to determine the zeros of a certain real valued function, we need to find the intervals in which this function changes sign. This method, however, can't be applied to complex valued functions. Thus we need to find a function that has the same zeros of the zeta function but which is real valued on the critical line. Luckily, such function does exist without any search and its $\xi(s)$. It was shown that this function is real valued. We will determine the simple zeros of this function by studying its sing changes. We develop Hardy's function following [3] :

$$\xi\left(\frac{1}{2}+it\right)=\left(\frac{1}{4}+i\frac{t}{2}\right).\left(-\frac{1}{2}+it\right).\pi^{-\frac{1}{4}-i\frac{t}{2}}.\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right).\zeta\left(\frac{1}{2}+it\right)$$

$$=-\frac{1}{2}\left(\frac{1}{4}+t^2\right).\pi^{-\frac{1}{4}-i\frac{t}{2}}.\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right).\zeta\left(\frac{1}{2}+it\right)$$

$$=\left[e^{Re\left(\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right)}.\pi^{-\frac{1}{4}}.\left(-\frac{t^2}{8}-\frac{1}{4}\right)\right].\left[e^{i\mathrm{Im}\left(\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right)}.\pi^{-\frac{it}{2}}.\zeta\left(\frac{1}{2}+it\right)\right]$$

since the brackets on the right is always a negative number, the sign of the Xi function always depends on the sign of the factor in the next set of brackets. By that we define Hardy's function to be:

$$Z(t):=e^{i\theta(t)}.\zeta\left(\frac{1}{2}+it\right)$$

Where:

$$\theta(t)=\mathrm{Im}\left(\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right)-\frac{t}{2}\ln\pi$$

From this point of view it seems that we haven't done anything because the zeta function also appears in the definition of Hardy's function. But this is not a problem, since by using the Riemann-Siegel formula (which we are going to present later on) a good approximation of this function will allow us to locate the zeros of the zeta function.

3.3 The Riemann Siegel formula

The Riemann Siegel formula is a very efficient method to calculate $\zeta(1/2+it)$. Let's recall the approximate functional equation of the zeta function. Suppose that $0<\sigma<1$ and suppose that $x,y,t \geq c>0$ and $2\pi xy=t$. Then uniformly on σ we find:

$$\zeta(s)=\sum_{m\leq x}\frac{1}{m^s}+\chi(s).\sum_{m\leq y}\frac{1}{m^{1-s}}+O(x^{-\sigma})+O(t^{1/2-\sigma}.y^{\sigma-1})$$

where $\chi(s)=2^s.\pi^{s-1}\Gamma(1-s).sin\,\pi s/2$. This equation can help us approximate Hardy's function in the following way. If we took $x=y=\sqrt{|t|/2\pi}$ in the approximate functional equation one finds that:

$$\zeta(s)=\sum_{n=1}^{[x]}\frac{1}{m^s}+\chi(s).\sum_{m=1}^{[x]}\frac{1}{m^{1-s}}+E_m(s)$$

Here the error term or $E_m(s)$ satisfies $E_m(s)=O(|t|^{-\sigma/2})$. If we placed $s=1/2+it$ we find upon multiplying by $e^{i\theta(t)}$:

$$Z(t)=2.\sum_{n=1}^{[x]}\frac{\cos\theta(t)-t\ln n}{\sqrt{n}}+O(t^{-\frac{1}{4}})$$

This is the basis of the Riemann Siegel formula. All what we need is to compute the error term with a big degree of accuracy. So the formula then becomes:

Theorem 1: for all $t\in\mathbb{R}$

$$Z(t)=2.\sum_{n=1}^{N}\frac{\cos\theta(t)-t\ln n}{\sqrt{n}}+\frac{e^{i\theta(t)-t\pi/2}}{2\pi^{\frac{1}{2}+it}.e^{-i\pi/4}(1-ie^{-t\pi})}.\int\frac{(-x)^{-\frac{1}{2}+it}}{e^x-1}.e^{-Nx}dx$$

where the integration is taken to be over the positively oriented closed contour C_n, which contains the points $\mp 2\pi iN, \mp 2\pi i(N-1),\dots,\mp 2\pi i$ and 0.

In practice, we often use a numerical approximation of the integral in order to find a useful way approximate $Z(t)$ and hence $\zeta\left(\frac{1}{2}+it\right)$. [16]

[16] Pugh, G.R. *The Riemann-Siegel formula and large scale computations of the Riemann zeta function*, page: 21.

3.4 Gram's law

Let's recall the following relation:

$$\zeta\left(\frac{1}{2}+it\right)=e^{-i\theta(t)}.Z(t)=\cos\theta(t).Z(t)-i\sin\theta(t).Z(t)$$

Gram in his computational notes, realized that the real part of $\zeta(1/2+it)$ tends to be positive. He then showed that the zeros of $\sin\theta(t)$ can be rather easily and he called them gram points. Formally, the n^{th} gram point is defined to be $\theta(g_n)=n\pi$. This definition leads to the formulation of Gram's law.

Theorem 2 (Gram's law): Hardy's function satisfies the following relation

$$(-1)^n Z(g_n)>0$$

Although it was named a law, Gram's law was proven to fail at infinity i.e it's not satisfied for all Gram points. But the importance of such law is that we can find a zero of the zeta function between any too gram points that satisfy the relation. We call the Gram points which satisfy Gram's law good Gram points, and those which don't bad gram points. We now give the definition of Gram blocks of order k. A Gram block of order k, is an interval of the shape $[g_n, g_{n+k}[$ where both the endpoints are good Gram points and all the Gram points between them are bad Gram points. A certain law concerned with these blocks is called Rosser's law.

Rosser's rule: A Gram block $[g_n, g_{n+k}[$ satisfies Rosser's rule if Hardy's function has at least k zeros in this Gram block.

Rosser's rule was proved to fail infinitely many times by Lehman. However, it is still useful in counting the zeros of the zeta function on the critical line. The following theorem is of great interest in counting the zeros of the zeta function. [17]

Theorem 3: if K consecutive Gram blocks whose union is $[g_n, g_p[$ satisfy Rosser's law where:

$$K\geq 0.0061\left(\ln g_p\right)^2+0.08.\ln g_p$$

then:

$$N(g_n)\leq n+1\ \ and\ p+1\leq N\left(g_p\right)$$

[17] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, pages: 34 and 35.

Chapter four: Equivalences, extensions, and consequences of the hypothesis

4.1 Equivalences to the hypothesis

In this chapter I shall mention the best and easiest equivalences to the hypothesis. And to be exact I'm going to talk about the Lagarias equivalent , an equality due to Volchkov in terms of a double integral, and some other analytic equivalences of the hypothesis.

Equivalence 1: The Riemann hypothesis is equivalent to the following double integral relation:

$$\int_{0}^{\infty} \frac{1-12t^2}{(1+14t^2)^3} dt \int_{1/2}^{\infty} \ln|\zeta(\sigma+it)| d\sigma = \pi \frac{3-\gamma}{32}$$

Proof :

let $\rho = \beta + i\gamma$ be a zero in the critical layer. It is known that:

$$\sum_{\gamma>0} f_\gamma(\beta) = \frac{\gamma}{4} + \frac{1}{2} + \frac{\ln 4\pi}{4} \quad , \quad f_\gamma(\beta) = \frac{\beta}{\beta^2+\gamma^2}$$

Since this function f is convex on the interval between 0 and 1, we have

$$f_\gamma(\beta) + f_\gamma(1-\beta) \leq 2f_\gamma(1/2)$$

Since the zeros ρ are symmetric with respect to the critical line, this fact and the equality first mentioned we find that the Riemann hypothesis is equivalent to the equality:

$$\sum_{\gamma>0} f_\gamma(1/2) = \frac{\gamma}{4} + \frac{1}{2} + \frac{\ln 4\pi}{4}$$

It was found that the sum on the left is equal to $A = \int_0^\infty f_x\left(\frac{1}{2}\right) dN(x)$ where $N(x)$ is the number of zeros of the zeta function that lie in the critical strip up to a height x. Using the estimate $N(x) = O(x.\ln x)$ for large x we have that:

$$A = \int_0^\infty N(x).g(x)dx$$

where:

$$g(x) = 16x/(1+4x^2)^2$$

It is also known that:

$$N(x) = 1 - (x.\ln \pi/\pi) + \mathrm{Im}(\ln \Gamma(1/4 + i\,x/2))/\pi + S(x)$$

where $S(x)$ is the increment of the argument of the zeta function along the broken line with vertices $s = 2, s = 2 + ix$ and $s = 1/2 + ix$. Then, $A = 2 - \ln \pi/4 + I_1 + \mathrm{Im}(I_2/\pi)$ where:

$$I_1 = \int_0^\infty S(x).g(x)dx$$

and:

$$I_2 = \int_0^\infty \ln \Gamma\left(\frac{1}{4} + i\frac{x}{2}\right).g(x)dx$$

Integrating by parts and using the formula:

$$\frac{\acute{\Gamma}(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n}\right)$$

we can easily calculate the integral I_2.We find that:

$$I_2 = -\frac{\gamma}{4} - \frac{\ln 2}{2}$$

Furthermore, if we integrate I_1 by parts we get:

$$S_1(x) = \int_0^x S(t)dt = O(\ln x)$$

we obtain that:

$$I_1 = -\int_0^\infty S_1(x)dg(x)$$

Thus we obtain the result wanted. [18]

[18] Volchkov, V. *On an equality equivalent to the Riemann hypothesis*, pages: 1,2 and 3.

To present the next equivalent we need the following lemma which is due to Balazard, Sais, and Yor:

Lemma 1: The Riemann zeta function satisfies the following relation:

$$\int_{1/2-i\infty}^{1/2+i\infty} \frac{\ln|\zeta(s)|}{|s|^s}|ds| = 2\pi \sum_{\beta>1/2} \ln\left|\frac{\rho}{1-\rho}\right|$$

where $\rho = \beta + i\gamma$ is the standard notation for the nontrivial zeros of the zeta function and the integral is taken over the line $Re(s) = 1/2$. The sum on the right hand-side is taken over all the nontrivial zeros of the function in which their real part is bigger than 1/2 which by the Riemann hypothesis do not exist meaning that the integral will be zero. This can be reformulated by the following equivalence:

Equivalence 2: The Riemann hypothesis is equivalent to the following relation:

$$\int_{1/2-i\infty}^{1/2+i\infty} \frac{\ln|\zeta(s)|}{|s|^s}|ds| = 0$$

I'm not going to mention the proof of the lemma so that I can leave some space for other equivalents. For a full proof of lemma 1 I refer the reader to [19]. The third equivalence is due to Lagarias which is considered one of the elementary equivalences to the hypothesis:

Equivalence 3: The Riemann hypothesis is equivalent the following statement :

$$\sigma(n) \leq H_n + e^{H_n} \ln H_n$$

where the function on the left hand-side is the sum of divisors function and H_n is the n^{th} harmonic number of order 1. [20] [6]

The next equivalence depends on the Fourier transform and its one of the best equivalences in attacking the hypothesis. It's due to Newman. But first of all let's put on some foundation. Let's define the following function:

$$\Upsilon(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-z/2-1/4}\,\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right)$$

It was shown that the function follows the relation:

$$\frac{1}{8}\Upsilon\left(\frac{z}{2}\right) = \int_0^\infty \Phi(t)\cos zt\,dt$$

where:

[19] Bui, H., S. Lester, and M. Milinovich, *On Balazard, Saias, and Yor's equivalence to the Riemann Hypothesis*, page: 1.

[20] Lagarias, J.C. *An elementary problem equivalent to the Riemann hypothesis*, page: 1.

$$\Phi(t)=\sum_{n=1}^{\infty}(2\pi^2n^4e^{9t}-3\pi n^2e^{5t})e^{-\pi n^2e^{4t}}$$

Let's define the family $H(s,\lambda)$ of functions parameterized by λ given as the following Fourier transform:

$$H(s,\lambda)=\int_0^{\infty}\Phi(t)e^{\lambda t^2}\cos st\,dt$$

We can see that when the parameter takes the value zero, the upper relation reduces to relation 6. Newman obtained some great properties of this family of functions which are:

1. for $\lambda \geq 1/2$ the function $H(s,\lambda)$ has only real zeros.
2. if the function has only real zeros for a certain constant λ', then it has only real zeros for all constants $\lambda > \lambda'$.

Newman (in a separate study in 1976) proved that there exists a parameter Λ such that the earlier function has at least one non-real zero and by that it is clear that the earlier function has only real zeros if $\lambda > \Lambda$. It shall also be clear that the Riemann hypothesis is equivalent to the saying that the earlier function "H" has only real zeros when the constant is equal to zero which implies that the constant Λ (which is called the de-bruijin-Newman constant) shall be smaller than zero. And by that I present the following equivalence:

Equivalence 4: The Riemann hypothesis is equivalent to the statement that the Newman constant is smaller than zero.

Several calculations were done to give lower bounds for this constant. The best result was due to Odlyzko who stated that the constant satisfies the following lower bound:

$$-2.7\times10^{-9}<\Lambda$$

and because of that he said: ".... the Riemann hypothesis if true, is just barely true". [21]

Now we are going to mention some analytic equivalences of the hypothesis. The first equivalence is concerned with the eta function:

Equivalence 5: The Riemann hypothesis is equivalent to the statement that all the zeros of the Dirichlet-eta function :

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

that fall in the critical strip lie on the critical line.

---

[21] Cislo, J. and Wolf,M. *Criteria equivalent to the Riemann Hypothesis,* pages: 3 and 4.

Equivalence 6: The convergence of:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $Re(s) > 1/2$ is a necessary and sufficient condition for the truth of the Riemann hypothesis.

Equivalence 7: The Riemann hypothesis is equivalent to the statement that the derivative of the zeta function is non-vanishing in the region $0 < Re(s) < 1/2$.

Now we are going to mention Li's criterion which is concerned with the positivity of a sequence of numbers defined by the following relation.

Equivalence 8: The Riemann hypothesis is equivalent to the positivity of every term in the following sequence:

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} [s^{n-1}. \ln \xi(s)]_{s=1}$$

I shall mention that the proof is so long that's why I couldn't write it down in the research. For more information about the proof I refer the reader to [22]. Other shapes of the above sequence also occur. In what follows I shall mention two of them.

Equivalence 9: The Riemann hypothesis is equivalent to the positivity of every term in the following sequence:

$$\lambda_n = \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right)$$

Equivalence 10: The Riemann hypothesis is equivalent to the positivity of every term in the following sequence:

$$\lambda_n = -n \sum_{j=1}^{n} \frac{(-1)^j}{j} . \binom{n+j-1}{2j-1} Z(j)$$

where:

$$Z(\sigma) = \sum_{k=1}^{\infty} {x_k}^{\sigma}$$

and $x_k = \frac{1}{4} + {t_k}^2$ and t_k is the imaginary part of the k^{th} non-trivial zero of the zeta function. [23]

[22] Li, X.-J. *The positivity of a sequence of numbers and the Riemann hypothesis*, page: 325 to 334.

[23] Voros, A. *Sharpenings of Li's criterion for the Riemann hypothesis*, pages: 2 and 3.

4.2 Extensions of the Riemann hypothesis

The Riemann zeta function is the main object of the Riemann hypothesis. However, this function is considered a special case of a wider family of functions called the L functions. We will discuss these functions and build them in a similar way to our build of the complete Riemann zeta function. We define a Dirichlet L series by the relation:

$$L(s,\chi_k) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s}$$

Where $\chi_k(n)$ is a number theoretic character defined by the relations:

$$\chi_k(1) = 1$$

$$\chi_k(n) = \chi_k(n+k)$$

$$\chi_k(m)\chi_k(n) = \chi_k(n.m)$$

for all integers m and n .we also have:

$$\chi_k(n) = 0$$

for $(k,n) \neq 1$. A character modulo k is said to be primitive if no $\chi_d(n)$ exists such that $\chi_k(n) = \chi_d(n)$ where d divides k and $d \neq k$. The unique character $\chi_k(n)$ such that $\chi_k(n) = 1$ for all n such that $\chi_k(n) = 1$ whenever $(k,n) = 1$ is called the principal character modulo k.

A Dirichlet L function $L(s,\chi_k)$ is defined to be the analytic continuation of the Dirichlet series. We can easily see that the Riemann zeta function lies within this group since:

$$L(s,\chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

We now state the Generalized Riemann hypothesis:

The Generalized Riemann hypothesis: all the non-trivial zeros of the function $L(s,\chi_k)$ have real part equal to 1/2.

By non-trivial we of course mean the zeros that satisfy the relation $0 < Re(s) < 1$. As we can see, the generalized Riemann hypothesis implies the Riemann hypothesis since the Riemann zeta function is a member of the family of L functions. We also note that these L functions, unlike the function $\zeta(s)$, may have zeros on the line $\mathrm{Im}(s) = 0$; however, all of these zeros are known and are considered trivial zeros.

Let p be an odd prime, we are going to define the Legendre symbol $\left(\frac{n}{p}\right)$ as follows. It takes the value 0 if the prime number divides the number n. It takes the value 1 if the congruence $x^2 \equiv n \pmod{p}$ has a solution, and the value -1 of it does not have any solution. Let us consider the series of the shape:

$$L_p(s) = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) . n^{-s}$$

The extended Riemann hypothesis is considered with the extensions of the series $L_p(s)$ to the whole complex plane and it states that:

The Extended Riemann hypothesis: all the zeros of $L_p(s)$ in which $0 < Re(s) < 1$, lie on the line $Re(s) = 1/2$.

Since the Legendre symbol is just an example of a number theoretic character, the extended Riemann hypothesis is an instance of the generalized Riemann hypothesis.

We are now going to mention an equivalent to the extended Riemann hypothesis. So, let's define the function $\pi(x, k, l)$ to be:

$$\pi(x, k, l) := \#\{p : p \leq x, p \text{ is prime, and } p \equiv l \bmod k\}$$

An equivalent to the extended Riemann hypothesis: for $(k, l) = 1$ and $\varepsilon > 0$:

where $\phi(k)$ is the Euler's totient function, and $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$.

Now we are going to mention another extended Riemann hypothesis. Let K be a number field with ring of integers $\mathcal{O}_k$. The Dedekind zeta function of the field K is given by:

$$\zeta_K(s) := \sum_{\alpha} N(\alpha)^{-s}$$

for $Re(s) > 1$, where the sum is over all integral ideals of the ring $\mathcal{O}_k$, and $N(\alpha)$ is the norm of α. Again we consider the continuation of these functions to the whole complex plane via analytic continuation.

Another extended Riemann hypothesis: All the zeros of the Dedekind zeta function of any algebraic number field which satisfy $0 < Re(s) < 1$, lie on the critical line.

This statement includes the Riemann hypothesis since the Riemann zeta function is the Dedekind zeta function over the field of rational numbers. [24]

[24] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, pages: 56,57 and 58.

Selberg's class and the Grand Riemann hypothesis:

We say that a function F is in the Selberg class $\mathcal{S}$ if it satisfies the following axioms:

1. In the half plane $\sigma > 1$ the function F is given by the convergent Dirichlet series:

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

   where $a(1) = 1$ and $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.
2. There is a natural number m such that $(s-1)^m F(s)$ is an analytic function in the whole complex plane. [25]
3. There is a function $\Phi(s) = Q^s . G(s) . F(s)$ where $Q > 0$ and:

$$G(s) = \prod_{j=1}^{r} \Gamma(\lambda_j . s + \mu_j)$$

   where $\lambda_j > 0$ and $Re(\mu_j) \geq 0$ such that:

$$\Phi(s) = \omega . \bar{\Phi}(1-s)$$

   where $|\omega| = 1$ and for any function f we denote $\bar{f}(s) = \overline{f(\bar{s})}$. Let's define d_F to be:

$$d_F = 2 \sum_{j=1}^{r} \lambda_j$$

   and we call it the degree of F.
4. We may express the logarithm of the function F by the following Dirichlet series:

$$\ln F(s) = \sum_{n=2}^{\infty} \frac{b(n) . \Lambda(n)}{n^s . \ln n}$$

   where $b(n) \ll \vartheta$ where $\vartheta < 1/2$. Set $b(n) = 0$ if n is not a prime power.

5. Also these functions satisfy what we call the Euler product:

$$F(s) = \prod_{p} F_p(s)$$

where $F_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying the condition $b(p^k) = O(p^{k\vartheta})$ where $\vartheta < 1/2$. The factors $F_p(s)$ are called the local factors. [26]

We denote by $\mathcal{S}_d$ the set of functions of $\mathcal{S}$ with degree d. It was found that $\mathcal{S}_0 = \{1\}$ and that $\mathcal{S}_d = \emptyset$ for $0 < d < 1$ and $\mathcal{S}_1 = \{\zeta(s) , L(s + iA, \chi)$. The latest result found concerned with the nature of the Selberg class is that $\mathcal{S}_d = \emptyset$ for $1 < d < 2$. The trivial zeros are:

---

[25] Perelli, A. *A survey of the Selberg class of L-functions,* pages: 83 and 84.

[26] Kaczorowski,J. and Perelli, A. *On the structure of the Selberg class, I: 0≤ d≤ 1*, pages: 1397 and 1398.

$$\rho = -\frac{\mu_j + k}{\lambda_j}$$

with $k \in \mathbb{N}$ and with $1 \leq j \leq r$. It was also shown that the non-trivial zeros of these functions in the Selberg class, all lie in the critical strip. So, an analogue of the Riemann – Von Mangoldt formula was found to count the zeros of any Selberg function up to a certain height T in the critical strip and this formula is:

$$N_F(T) = \frac{d_F}{\pi} T \ln T + c_F . T + O(\ln T)$$

where d_F is the degree of the function F and c_F is a certain constant that depends on the function. It was also conjectured that the functions in the Selberg class satisfy an analogue of the Riemann hypothesis called the Grand Riemann hypothesis which is considered to be the largest extension of the Riemann hypothesis and it states that:

The Grand Riemann hypothesis: All the non-trivial zeros of every function in the Selberg class lie on the line $\sigma = 1/2$. [27]

I shall note that many similar properties to the Riemann zeta function exist for these functions but we shall not mention them because they are out of the scope of this research.

[27] Perelli, A. *A survey of the Selberg class of L-functions,* pages: 107 and 108.

4.3 Consequences of the truth of the Riemann hypothesis

Proving the Riemann hypothesis will not only create a great havoc in the world of mathematics, it will also prove hundreds and hundreds of other related problems. In this section we talk about some of the most famous consequences of the truth of the Riemann hypothesis. First of all we start with the prime number theorem.

Theorem 1 (The PNT): the number of prime numbers less than x is given by the formula:

$$\pi(x) = Li(x) \approx \frac{x}{\ln x}$$

where $Li(x) = \int_2^x \frac{dt}{\ln t}$.

A concise and a sufficient sketch of the proof will be given in what follows, but first of all, we shall present what we call Chebyshev's counting function:

$$\psi(x) = \sum_{p^k \leq x} \ln p$$

the two counting functions $\psi(x)$ and $\pi(x)$ are connected by the following tremendous relation:

$$\pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) \ldots = \int_2^x \frac{d\psi(t)}{\ln t}$$

This gives two equivalent statements to the PNT namely:

$$\pi(x) \approx Li(x) \leftrightarrow \psi(x) \approx x$$

Riemann found what we call the explicit formula (proved by Von Mangoldt) which can be written as:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n} \frac{x^{-2n}}{2n} - \ln 2\pi$$

A part of Riemann's plan to prove the PNT was to be able to estimate the sum concerning the non-trivial zeros of the Riemann zeta function to conclude that $\psi(x) \sim x$. This great strategy actually worked. Since we know that there are no zeros on the line 1, we find that the following limit is satisfied:

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\rho} \frac{x^{\rho}}{\rho} = 0$$

This leads to that $\psi(x) \sim x$ which in turn implies that $\pi(x) \approx Li(x)$ [28]. Such a powerful bound was found just by implying that $\zeta(1+it) \neq 0$. So what will happen if it was true that all the zeros of the function lie on the critical line?. Actually, the following consequence of the Riemann hypothesis gives the whole story [29].

Theorem 2: :Let the set Z denote the set of non-trivial zeros of the zeta function. Suppose that there exists a number $\theta < 1$ such that:

$$Z \subset \{s \in \mathbb{C} : 1 - \theta \leq Re(s) \leq \theta$$

Then as $x \to \infty$ we have:

$$\psi(x) = x + O(x^{\theta}.(\ln x)^2)$$

In case the Riemann hypothesis is true we find the following consequence:

Consequence 1: If the Riemann hypothesis is true then we find that [30]:

$$\pi(x) = Li(x) + O(\sqrt{x}.(\ln x))$$

Next we are going to talk about Goldbach's conjecture. Goldbach conjectured, in a letter to Leonhard Euler, that every natural bigger than five can be written as the sum of three primes. This conjecture is considered to be one of the oldest unsolved problems in number theory. A related problem is the Goldbach's weak conjecture.

Goldbach's weak conjecture: Every odd number can be expressed as the sum of 3 prime numbers.

Hardy and Littlewood proved that the Riemann hypothesis implies the latter conjecture. In 1997 Deshouillers, Effinger, te Riele, and Zinoviev proved the following result.

Consequence 2: Assuming the generalized Riemann hypothesis, every odd natural number bigger than 5 can be expressed as the sum of three primes.

Later on, Hardy and Littlewood proved that under the assumption of the generalized Riemann hypothesis, nearly every even number can be written as the sum of two primes. An important result, which is also a consequence of the hypothesis, was given by J.J Chen in 1973 and is formulated as follows:

Consequence 3 (Chen's Theorem): every sufficiently large even integer can be written as the sum of a prime number and a number that is written as the product of at most 2 primes.

The Riemann hypothesis is also connected to Bertrand postulate which states that there always exists a prime number between a and $2a$. It is not difficult tot prove the following theorem under the assumption of the generalized Riemann hypothesis.

---

[28] David,B. *Riemann's Zeros and the Rhythm of the Primes,* pages: 17 to 27.
[29] Segarra, E. *An Exploration of … Application to the Theory of Prime Number Distribution,* page:50.
[30] Bombieri,E. *Problems of the Millennium: the Riemann Hypothesis,* page: 4.

Consequence 4: If the Riemann hypothesis is true, then for sufficiently large x and for any number $a > 1/2$ there exists a prime p in the interval $(x\,, x + x^a)$.

Now we are going to talk about the mean values of the Riemann zeta function. A moment or mean value of the Riemann zeta function $\zeta(s)$ is given by the relation:

$$I_k(T) = \frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt$$

for an integer $k \geq 1$. These mean values have a lot of interesting application in zero-density theorems for the Riemann zeta function $\zeta(s)$ and various divisors problems.

In 1918, Hardy and Littlewood proved that:

$$I_1(T) \sim \ln \mathrm{T}$$

as $T \to \infty$. In 1926, Ingham proved that:

$$I_2(T) \sim \frac{1}{2\pi^2}(ln\, T)^4$$

Although it had been conjectured that:

$$I_k(T) \sim c_k (ln\, T)^{k^2}$$

where c_k is a certain constant, no asymptotic formula for the mean values of the zeta function for $k \geq 3$ has been found yet. Conrey and Ghosh, in an unpublished work, found a more precise form of the constant c_k namely:

$$I_k(T) \sim \frac{a(k).\, g(k)}{\Gamma(k^2 + 1)}(ln\, T)^{k^2}$$

where:

$$a(k) = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} . \sum_{m=0}^{\infty} \left(\frac{\Gamma(m + k)}{m!\,\Gamma(k)}\right)^2 . p^{-m}\right)$$

and $g(k)$ is an integer whenever k is an integer. Conrey and Ghosh then developed the following consequence.

Consequence 5: The Riemann hypothesis implies the following result which is concerned with the mean values of the Riemann zeta function as $T \to \infty$ [31]:

$$I_k(T) \geq (a(k) + o(1))(ln\, T)^{k^2}$$

[31] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, pages: 62 , 66 and 67.

Chapter five: My equivalences

I shall mention that after an extensive study of the different equivalences of the hypothesis and after a tedious work with harmonic number and polygamma functions, I was able to derive an equivalence that I have never seen yet considered with the values of the zeta function at integer arguments. I derived my equivalence after studying the Lagarias equivalent to the Riemann hypothesis. The identities that I used to manipulate the Lagarias equivalent shall be mentioned below ,and of course I shall mention them without proofs.

The nth harmonic H_n is defined by the relation:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

The polygamma function of order n is defined by the following relation:

$$\psi_n(z) = \frac{d^{n+1}}{dz^{n+1}}(\ln \Gamma(z))$$

where the function under the derivative sign is the natural logarithm of the gamma function.

Now after stating the definitions I shall note that the zero order polygamma function is called the digamma function or the psi function and our main attention will be attached to it. I now state the relations used in the proof of the equivalence.

Lemma 1: The nth harmonic number and the digamma function satisfy the following equality relation:

$$H_n = \psi(n+1) + \gamma$$

where γ is the Euler constant defined as:

$$\gamma = \lim_{x\to\infty}\left(\sum_{n=1}^{x} \frac{1}{n} - \ln x\right)$$

Lemma 2: The following sum identity concerning the harmonic numbers holds for an integer number x [32]:

$$2 \times \sum_{n=1}^{\infty} \frac{H_n}{n^x} = (x+2)\zeta(x+1) - \sum_{r=1}^{x-2} \zeta(r+1)\zeta(x-r)$$

Lemma 3: The polygamma functions satisfy the recurrence relation:

$$\psi_n(z+1) = \psi_n(z) + (-1)^n n!\, z^{-n-1}$$

---

[32] Sofo , A. *Harmonic number sums in closed form*, pages: 1 and 2.

Lemma 4: Let $\sigma(n)$ be the sum of the divisors function. then the following relation holds [33]:

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n^x}=\zeta(x)\zeta(x-1)$$

Now we arrive to the grand theorem which can be stated as follows:

Theorem 1: The Riemann hypothesis is equivalent to the statement that for every integer x bigger than two the following inequality holds:

$$\zeta(x)\zeta(x-1)<\frac{x+2}{2}\zeta(x+1)-\frac{1}{2}\sum_{r=1}^{x-2}\zeta(r+1)\,\zeta(x-r)+e^{\gamma}.\sum_{n=1}^{\infty}\frac{e^{\psi(n+1)}.\ln(\psi(n+1)+\gamma)}{n^x}$$

Proof:

first of all let's remember Lagarias's equivalent which we mentioned earlier. The Riemann hypothesis is true if and only if the following inequality holds for all natural numbers n bigger or equal to one:

$$\sigma(n)\leq H_n+e^{H_n}.\ln H_n$$

where the function on the left hand-side of the inequality is the sum of divisors function. I attempted to manipulate the last equivalent to create another one containing the zeta function itself perhaps it could reveal to us some of the hidden properties of the function. My first step was that I divided by n^x and then sum from 1 to infinity. Then by applying lemmas 5 and 2 the inequality takes the following shape:

$$\zeta(x)\zeta(x-1)<\frac{x+2}{2}\zeta(x+1)-\frac{1}{2}\sum_{r=1}^{x-2}\zeta(r+1)\zeta(x-r)+\sum_{n=1}^{\infty}\frac{e^{H_n}.\ln H_n}{n^x}$$

by applying lemmas 1and 3 consecutively we get the wanted result.

As you can see the last inequality is still a bit messy. My next goal is to simplify the inequality by getting rid of the infinite sum on the right hand-side of it maybe by putting some assumptions on the number x. Right now, I'm going to mention the other equivalent that I found during my study and it depends on the following Lemma.

Lemma 6: The following double integral representation holds for the Riemann zeta function:

$$\iint\frac{(-\ln x.y)^s}{1-xy}dxdy=\Gamma(s+2).\zeta(s+2)$$

where the double integral is taken over the unit square [34].

---

[33] TITCHMARSH,E. *Theory of the Riemann zeta function*, page: 8.

[34] Borwein, P. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*, page: 78.

Theorem 2: The Riemann hypothesis holds true if for every positive number γ (which represents the ordinates of a zero of the zeta function) the following relation holds:

$$\iint \frac{(-\ln x.y)^{-3/2+i\gamma}}{1-xy} dxdy = 0$$

Proof:

By a simple substitution to the right hand side of lemma 6 we find:

$$\iint \frac{(-\ln x.y)^{-3/2+i\gamma}}{1-xy} dxdy = \Gamma(1/2+i\gamma).\zeta(1/2+i\gamma)$$

and the truth of the Riemann hypothesis implies that the last term on the right hand side of the equality is zero since all the zeros of the zeta function lie on the line 1/2.

Conclusions

We have talked about the Riemann hypothesis, its properties, extensions, and consequences which clarified and magnified the importance of this hypothesis and showed its importance and connections to many other problems in mathematics and not only simple problems but large ones like the problem of the prime numbers and Goldbach's conjecture and other problems which we mentioned earlier. We also showed and discussed the best and most famous equivalences of the hypothesis and said that those were the best and most effective equivalences to attack the hypothesis. I also found and derived two new equivalences of the hypothesis which I haven't seen anywhere and I'll keep working on them to reformulate them in a better and more effective way in my attempt to solve this tremendous hypothesis. Of course, the Riemann hypothesis is still standing on the top of the unsolved problems in mathematical history described as one of the most influential and important problems of the millennium, and maybe and just maybe I'll have the honor to solve such puzzling problem !!.

References

1. BOROWEIN,P.;CHOI,S.;ROONEY,B.;WEIRATHMUELLER,A. *The Riemann Hypothesis*. Springer, 2007 , 588.
2. BORWEIN,J.M.; BRADLEY,D.M.; CRANDALL,R.E. *Computational strategies for the Riemann zeta function*. USA, 1999, 68.
3. MENICI,L. *Zeros of the Riemann zeta function the critical line*. University of Rome, Italy, 2012, 90.
4. SEGGARA,E. *An exploration of the Riemann zeta function and its applications to the theory of prime number distribution.* 1st edition, Harvey Mudd College, 2006, 66.
5. CONREY,J.B. *More than two fifths of the zeros of the Riemann zeta function are on the critical line*. Journal für die reine und angewandte Mathematik, 1989, 26.
6. PUGH,G.R. *The Riemann Siegel formula and large scale computations of the Riemann zeta function.* University of New Brunswick, British Columbia, 1992, 82.
7. LAGARIAS,J.C. *The Riemann hypothesis: arithmetic and geometry*. 2007,16.
8. BOMBIERI,E. *Problems of the millennium: The Riemann hypothesis*. Institute of advanced study, Princeton, 2001,11.
9. KUZNETSOV,A. *On the Riemann Siegel formula*. University of New Brunswick, British Columbia, 2007, 11.
10. MINAC,J.; Dwilewicz,R.J. *Values of the Riemann zeta function at integers*. University of Barcelona, Spain, 2009, 26.
11. SEBAH,P.;GOURDON,X. *Distribution of the zeros of the Riemann zeta function*. 2004,6.
12. LAGARIAS,C.J. *An elementary problem equivalent to the Riemann hypothesis*. USA,2001, 10.
13. Zudilin, W. *Irrationality of values of the Riemann zeta function*. American mathematical society (AMS), 2002, 54.
14. GHUSAYNI,B. *Results connected to the Riemann hypothesis*. Department of mathematics Hadath university, Lebanon, 2012, 16.
15. CHRIST,T. *Value distribution of the Riemann zeta function and related functions near the critical line*. 1st edition, Deutschland, 2014, 191.
16. LAURINCIKAS, A. *Universality of the Riemann zeta function*. Journal of Number theory, 2010, 9.
17. MAUCLAIRE, J.L. *Universality of the Riemann zeta function: two remarks*. France , 2012 , 9
18. VOLCHKOV, V. *On an equality equivalent to the Riemann hypothesis*. Ukrainian mathematical journal, 1995,3.
19. BUI, H., S. LESTER, and M. MILINOVICH. *On Balazard, Saias, and Yor's equivalence to the Riemann Hypothesis*. 2010,11.
20. CISLO, J.;M. WOLF. *Criteria equivalent to the Riemann Hypothesis.* ResearchGate ,2008,10.
21. LI, X.-J. *The positivity of a sequence of numbers and the Riemann hypothesis*. Journal of number theory, 1997, 9.

22. VOROS, A. *Sharpenings of Li's criterion for the Riemann hypothesis*. Service de Physique Theorique de Saclay, 2013,10.
23. PERELLI, A. *A survey of the Selberg class of L-functions.* American mathematical society, 2005,36.
24. KACZOROWSKI, J. and A. PERELLI. *On the structure of the Selberg class, I: 0≤ d≤ 1*. Annals of mathematics, 2011,45.
25. DAVID, B. *Riemann's Zeros and the Rhythm of the Primes*. Emory university,2009,28.
26. SOFO, A. *Harmonic number sums in closed form*. Department of mathematics university of Osijek, 2010,11.
27. TITCHMARSH, E. *Theory of the Riemann zeta function*. Oxford university press, 1986,211.
28. Steuding, J. *The universality of the Riemann zeta function*. Guwahati, 2008, 90.

Index of pictures